PC 30

3.1 Characteristics of Polynomial Functions

To identify polynomial functions and their degree, to identify end behavior and the possible number of x intercepts.

A polynomial function is a function of the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots a_2 x^2 + a_1 x^1 + a_0$ where:

- n is a whole number, x is a variable, and the coefficients a_n to a_0 are real numbers
- The ***degree*** **(n)** of a polynomial function is the ***greatest exponent*** of the variable x that exists in the equation of the function.
- The ***coefficient*** of the greatest power of x is the ***leading coefficient***. The text book calls this term a_n.
- The ***constant term*** represents the $y-$intercept of the function. The text book calls this term a_0.

NOTE: Names of Polynomials Functions

- Degree 0 – Constant Function
- Degree 1 – Linear function
- Degree 2 – Quadratic function
- Degree 3 – Cubic function
- Degree 4 – Quartic function
- Degree 5 – Quintic function

In general, the degree of a polynomial can predict how many times a graph runs through the x axis. Can you use this info to predict what a graph of each degree might look like? (Note that polynomial graphs always curve when they turn, they will never have sharp corners)

Example #1: Which of the following are polynomials? Explain why or why not. If they are polynomial functions, state the degree, the leading coefficient and the constant term.

a) $y = \sqrt{x} + 7$

b) $y = 8x^5 - 6x^{-2} + 5$

c) $y = 8x^5 - 6x^2 + 5$

d) $y = 6x^{\frac{1}{3}}$

e) $y = \frac{x}{5}$

f) $y = \frac{5}{x}$

g) $y = -2x^2 + 6x - 8$

h) $y = x$

i) $y = 7$

j) $x = -1$

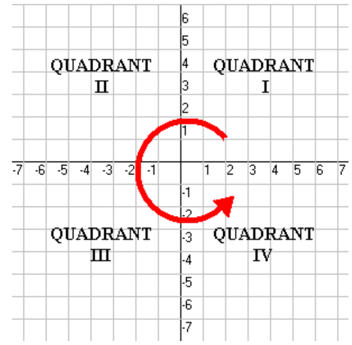
k) $y = 4^x$

l) $y = 8 - 9x^4 + 8x^7 + 12x^2$

m) $y = |x|$

END BEHAVIOR:

Graph the following graphs one at a time. We read graphs lik e we are reading a book – we start on the left side of the grid, get on the graph and “drive” across the graph until we reach as far right as we can go on the grid. Using this “driving” process, decide which quadrant you start driving on for each graph and which quadrant you end driving on.



EQUATION	IS EQUATION EVEN or ODD?	SKETCH	STARTING QUADRANT	ENDING QUADRANT	BEHAVES LIKE
$y = x^2$					
$y = -x^2$					
$y = x^4 - 5x^2 + 4$					
$y = -(x^4 - 5x^2 + 4)$					
$Y = x$					
$Y = -x$					
$Y = x^3 - 4x^2 + 4x - 1$					
$Y = -x^3 + x^2 + 4x - 1$					

SUMMARY: The degree and the leading coefficient will tell you which quadrant the polynomial function will start and end in. It will also give you an indication of the maximum possible number of x intercepts. Fill in the following chart to summarize what we have found:

LEADING COEFFICIENT	DEGREE	STARTS IN QUADRANT	ENDS IN QUADRANT	BEHAVES LIKE: Negative/Positive Line or Parabola	
+	EVEN				
-	EVEN				
+	ODD				
-	ODD				

Example #2:

For each polynomial function, identify the following characteristics:

- the type of function and whether it is of even or odd degree
- the end behaviour of the graph of the function
- the number of possible x-intercepts
- whether the graph has a maximum or minimum value
- the y-intercept

Then, match each function to its corresponding graph.

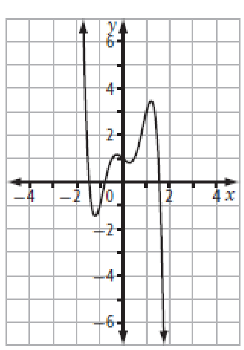
a) $f(x) = 2x^3 - 4x^2 + x + 2$

b) $g(x) = -x^4 + 10x^2 + 5x - 6$

c) $h(x) = -2x^5 + 5x^3 - x + 1$

d) $p(x) = x^4 - 5x^3 + 16$

A



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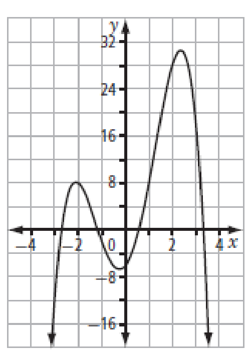

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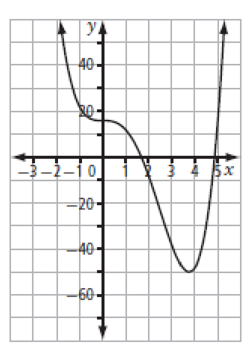

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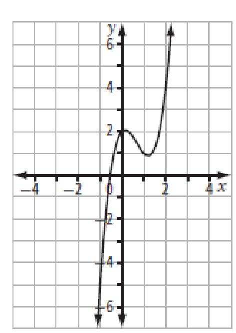


NOW: GO BACK INTO THE TABLE AT THE TOP OF THIS PAGE AND FILL IN WHICH TYPES WILL HAVE MAXIMUMS AND WHICH TYPE WILL HAVE MINIMUMS.

Example #3:

An antibacterial spray is tested on a bacterial culture. The population, P, of bacteria t minutes after the spray is applied is modelled by the function $P(t) = -2t^3 - 2t^2 + 3t + 800$.

a) What is the population of the bacteria 3 min after the spray is applied?

b) How many bacteria were in the culture before the spray was applied?

c) What is the population of the bacteria 8 min after the spray is applied? Why is this not realistic for this situation? Explain.

3.1 ASSIGNMENT

3.1 FA: P114 ##1-4, 6

3.1 MLA: P114 # C2 (p116), 7, 9, 12

3.1 ULA: P114 # 5, 8, 11, 13

PC 30

3.2 The Remainder Theorem

To use long division and synthetic division to divide a polynomial by a binomial and to find the remainder.

FLASHBACK TO ELEMENTARY SCHOOL:

Divide 327 by 12 using long division.

•What is the proper way to write the remainder?

•How can you check your answer?

Example #1: We will now divide a polynomial, P(x), by a binomial (x – a)

$\left(x^2+7x+17\right)\div\left(x+3\right)$

STEPS:

a) State the restrictions on the variable.
b) Write the polynomial in order of descending powers.
c) Do long division ☺
d) Write your final answer in the form:

$$\frac{P(x)}{x-a}=Q(x)+\frac{R}{x-a}$$

e) Check!

Example #2: Divide the following:

$\left(5x^3+10x-13x^2-9\right)\div(x-2)$

Example #3:

The volume, *V*, in cubic centimetres, of gift boxes is given by $V(x) = 2x^3 + x^2 - 27x - 36$. The height, *h*, in centimetres, is $x + 3$. What are the possible dimensions of the boxes in terms of *x*?

- Long division helps us factor polynomials that can't be factored using our grade 11 methods (decomposition, window method, trial and error, difference of squares etc)
- Unfortunately long division is very cumbersome and silly mistakes are often made.
- There is a better way to accomplish long division by a process that is called SYNTHETIC DIVISION. Synthetic division allows us to get the same answer as long division by ignoring all the variables and taking up much less space!

SYNTHETIC DIVISION STEPS:

1. Draw a long "L" and place the coefficients of the polynomial within the "L"
2. If you consider your binomial in the form "x – a", then take "a" and place it outside of your "L" (Note: the sign on the value of a switches)
3. "Pull down" the first number, then multiply "a" by this number and place it in the second column
4. Add the two numbers in this column together and place the result underneath.
5. Multiply this number by "a", place it in the third column and repeat step 4
6. Continue until you are out of columns.
7. Write your answer in the form $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$

Let's try our first example using this method:

$(x^2 + 7x + 17) \div (x+3)$

Example #4: Use Synthetic division to divide the following:

a) $(2x^3 + 5x^2 - x - 6) \div (x+2)$

b) $x-3\overline{)3x^4 - x^3 - 5}$

c) Divide P(x)=$x^3 + 3x^2 - 2x + 5$ by (x + 1)

RULE: When a polynomial is written in terms of descending order of power of terms and there is a term missing, in order to use synthetic division you must ________________________________

QUESTION: What does it mean when the remainder is zero?

Example #2b: What remainder did we get when we divided the following earlier in example #2?

$\left(5x^3+10x-13x^2-9\right)\div(x-2)$ REMAINDER WAS:

Let me reword this question a bit. What if we said that $P(x)=5x^3+10x-13x^2-9$ and that $x=2$. Can you find $P(2)$? What do you notice about this answer and the remainder?

THE REMAINDER THEOREM:
If a polynomial P(x) is being divided by a binomial x – a, you can find the remainder without doing long division or synthetic division. Instead, to find the remainder just find P(a).

Example #5:
Use the remainder theorem to find the remainder when $11x-4x^4-7$ is divided by $x-3$. Verify using synthetic division.

3.2 ASSIGNMENT

3.2 FA: P124 #1, 2, 3ef, 4ace, 5ace, 6abf
3.2 MLA: P124 # 8ab, 10, 11, 12, 14, 17a and any of 3a-d, 4bdf, 5bdf, 6bdf
3.2 ULA: P124 # 13, 15, 16, 17bc, C2, C3

3.3 The Factor Theorem

To use the factor theorem to factor polynomials.

Remember: What does it mean to factor an expression?

Review: Factor the following polynomials:

a) 24

b) $2x + 4$

c) $4x^2 - 9y^2$

d) $x^2 + 3x - 4$

e) $2x^2 + 9x + 4$

f) $12x^2 - 22x - 20$

What is the difference between solving and factoring an expression?

Can you use any of the above methods to factor $2x^3 - 5x^2 - 4x + 3$?

- How many x intercepts will the above expression likely have if it was written as a function (with y = in front)?

- How many factors (sets of brackets) will the question likely factor into?

We will use what we learned about the remainder theorem yesterday to develop the FACTOR THEOREM which allows us to factor polynomials of degree$\geq$3!

THE FACTOR THEOREM:
If "a" is a zero of a polynomial P(x) by the remainder theorem, then (x – a) is a FACTOR of the polynomial. (Remember that we can test to see if "a" is a zero by testing to see if P(a)=0)

Example #1: Is $(x - 3)$ a factor of the polynomial $x^3 + 2x^2 - 5x - 30$?

Example #2: Is (2x + 1) a factor of the polynomial $2x^3 + 7x^2 -+7x + 2$?

Example #3: Fully Factor $2x^3 - 5x^2 - 4x + 3$

STEPS TO FACTORING :

1. Decide how many factors there will LIKELY be based on the degree of the polynomial and write down that number of brackets.
2. Begin by finding the first factor (x – a). The value of must divide into the constant term of the polynomial and P(a)=0. Make a list of all numbers (positive & negative) that divide into the constant term, test all of them until you find a value of "a" where P(a) = 0.
3. Perform synthetic division on the polynomial using the value of "a".
4. Rewrite the numbers you received as answers to the synthetic division as the coefficients/constant of a polynomial.
5. If the polynomial in step 4 is degree 2, factor it to obtain the remaining factors to fill into step 1. If the polynomial in step 4 is higher than degree 2, return to step 2 and continue to repeat steps 2-4 until you obtain a degree two polynomial.
6. For your final answer, rewrite the original polynomial and the factors it is equal to at the bottom of the question.

Example #4: Fully Factor the following:

a) $x^4 - 3x^3 - 7x^2 + 15x + 18$

b) $x^4 - 5x^3 + 2x^2 + 20x - 24$

.

3.3 ASSIGNMENT

3.3 FA: P133 #1ac, 2abef, 3ae, 4ade, 5ace, 6ace

3.3 MLA: 7, 8, 10, 11 & additional questions from 1-6

3.3 ULA: P133 #12, 13, 14, 15, 16, C3

3.4 Equations & Graphs of Polynomial Functions

To sketch the graph of a polynomial function without the use of technology and model and solve problems involving polynomial functions.

MULTIPLICITY:

Refers to the number of times that a factor (x – a) repeats itself within the factoring a a polynomial P(x)

OR

Refers to the number of times that a zero/root/solution of a function "a" where P(a) = 0 occurs within a polynomial P(x)

- **RECALL:** When you find the factors of a polynomial what you are finding is the **zeroes** or the **x-intercepts** of the function.
- **The multiplicity of a root/zero/solution of a function affects the path the graph takes as it crosses the x axis at that corresponding x intercept**

Example #1: Given the function $f(x) = (x - 1)(x + 2)^2(x - 4)^3$ describe the multiplicity of each of the factors and describe what the x intercepts will be.

CASE 1: Factors with a Multiplicity of One

1. Use technology to sketch the following polynomial functions with factors that each have a multiplicity of one.
   a) $y = (x - 4)$ b) $y = -2(x + 2)(x - 1)$ c) $y = x(2x - 1)(x + 6)$

2. What (if anything) do you notice about how the graph crosses through each x intercept?

CASE 2: Factors with a Multiplicity that is an EVEN Number (2, 4, 6,.....)

3. Use technology to sketch the following polynomial functions with factors that each have an even multiplicity.
   a) $y = (x - 4)^2$ b) $y = -2(x + 2)^2(x - 1)^4$ c) $y = (x - 3)^6$

4. What (if anything) do you notice about how the graph crosses through each x intercept?

CASE 3: Factors with a Multiplicity that is an ODD Number >1 (3, 5, 7,.....)

5. Use technology to sketch the following polynomial functions with factors that each have an odd multiplicity.

 a) $y = (x - 4)^3$ b) $y = -2(x + 2)^5(x - 1)^3$ c) $y = x^7$

6. What (if anything) do you notice about how the graph crosses through each x intercept?

GRAPHING POLYNOMIAL FUNCTIONS: Following are the techniques we will use together to sketch the graph of a polynomial function P(x)

1. Start/End behavior chart

LEADING COEFFICIENT	DEGREE	STARTS IN QUADRANT	ENDS IN QUADRANT	BEHAVES LIKE
+	EVEN	2	1	+ Parabola
-	EVEN	3	4	- Parabola
+	ODD	3	1	+ Line
-	ODD	2	4	- Line

2. Factoring the polynomial using the remainder and factor theorems and synthetic division
3. How the graph crosses the x-axis by looking at each factor and their multiplicity

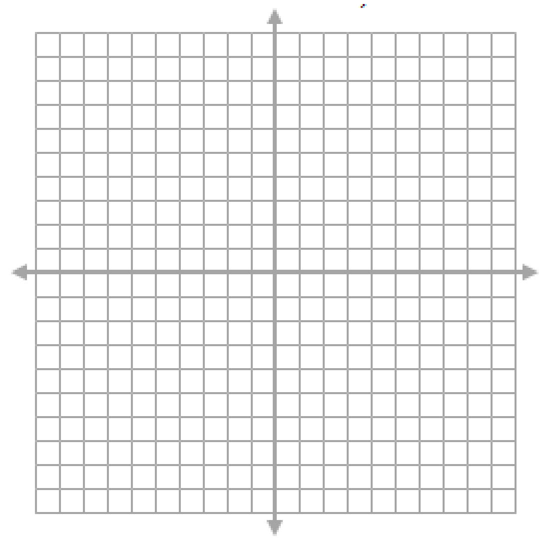
Multiplicity	STYLE OF CROSSING X AXIS
One	Crosses
Even	Tangent to
Odd	Tangent to and crosses

4. The y intercept found by calculating the value of P(0)
5. Using Sign Analysis (NEW) which determines when the graph of the function is above the x axis $P(x)>0$ and where the graph of the function is below the x axis $P(x)<0$

Example #2: Use sign analysis to determine the intervals where the function is positive and the intervals where the function is negative (this means to find where the graph of the following function is above and where it is below the x axis) $f(x) = (x - 1)(x + 2)^2(x - 4)^3$

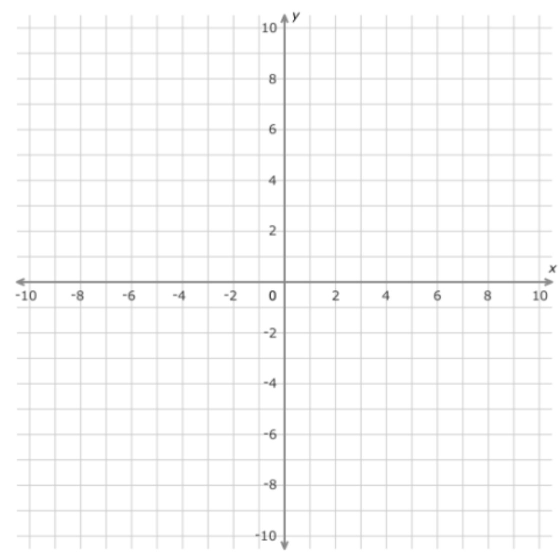
Example #3: Using the sign analysis and the function in Example 2 as well as the 5 graphing techniques found on the previous page, sketch the graph of the function

$f(x) = (x - 1)(x + 2)^2(x - 4)^3$

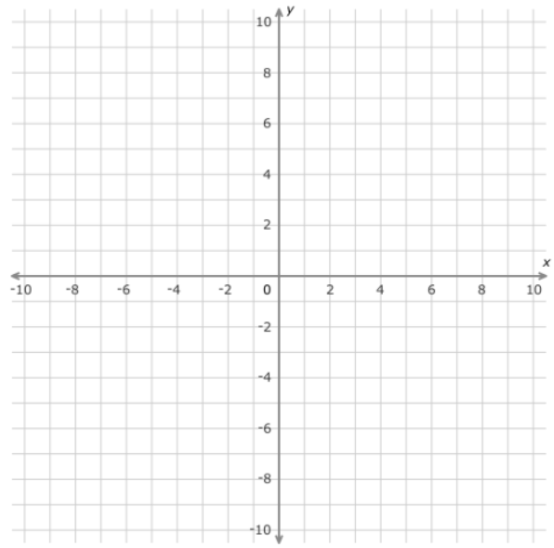


Example #4: Without using technology, use the 5 techniques listed to sketch the following polynomial functions.

a) $f(x) = 2x^3 - 3x^2 - 5x + 6$

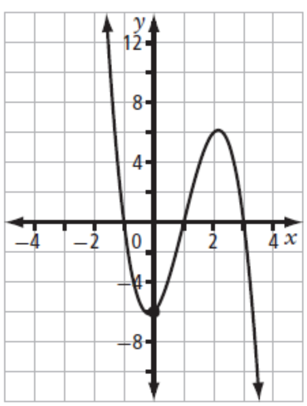


b) $g(x) = -x^3 - 5x^2 - 3x + 9$

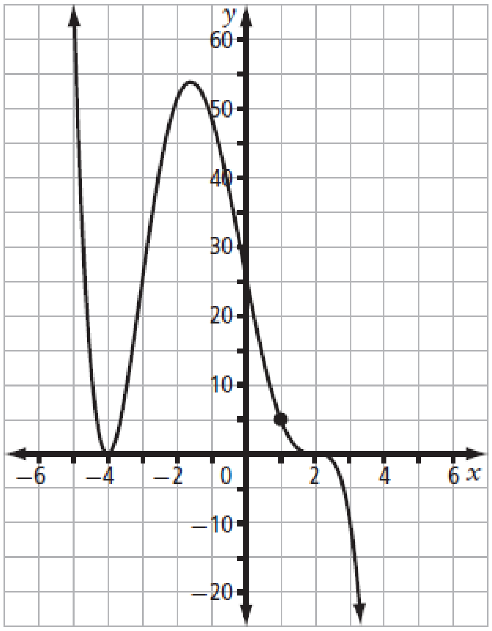


Example #5: Use the graph of the given functions to write the possible corresponding polynomial equation. Write the intervals where the function is positive and where it is negative. How can you determine if there is a coefficient (this is a coefficient that is factored out during the synthetic division stage and is in front of the factors?

a)



b)



Example #6:

An interlocking stone path that is *x* feet wide is built around a rectangular garden. The garden is 20 ft wide and 40 ft long. The combined surface area of the garden and the walking path is 1196 ft2. What are the dimensions of the stone path?

3.4 ASSIGNMENT (Must use loose graph paper):

3.4 FA: P147 #1ac, 2, 3, 4, 5, 7b, 9ade, 10ad

3.4 MLA: P147 #13, 14, 15, 16, 6 and additional questions from 1-5, 7, 9 & 10

3.4 ULA: P149 #11, 17, 18, 20, 21, 23

VIDEO LINKS THAT MAY AIDE IN UNDERSTANDING

Section 3.1

- https://goo.gl/dXvoqs
- https://goo.gl/PGdeJ9

Section 3.2 Polynomial Functions: The Remainder Theorem

- https://goo.gl/7iSi4C
- https://goo.gl/fnP9j8

Section 3.3 Polynomial Functions: Factoring

- https://goo.gl/HnTxmp
- https://goo.gl/7NJgek

Section 3.4 Graphing Polynomials:

- https://goo.gl/tG7663
- https://goo.gl/E9aAJH