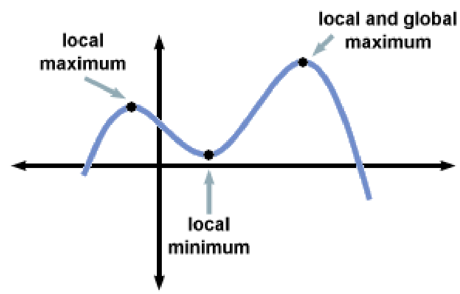
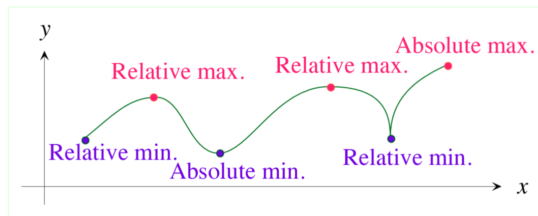
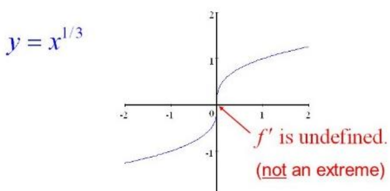
**To learn to use the first derivative to find maximum and minimum values of a function and to use the First Derivative Test to find increasing and decreasing intervals of a function.**

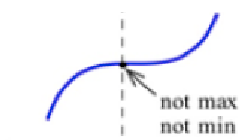


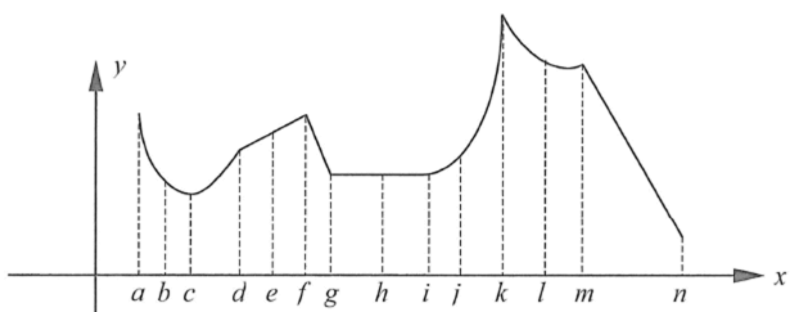

A graph of a function may have a series of highs and lows.

- The highest point is called the **Global or Absolute Maximum**
- The lowest point is called the **Global or Absolute Minimum**
- Together, the lowest and highest points are called the **Global or Absolute EXTREMA.**
- In general, every high point is called a **Local or Relative Maximum** if it is higher than all function values close to it
- In general, every low point is called a **Local or relative Minimum** if it is lower than all function values close to it
- Together, all high and low points are called **Local or Relative Extrema**
- All maximum and minimum points occur at places in the interior of the graph where either the tangent line to the graph has a slope of zero, or where the tangent line does not exist.
  - **CRITICAL POINTS** on the graph are places in the interior of the graph where local extrema exist on the graph of the function. These points, x=c, exist where either f'(c) = 0 or f'(c) does not exist (being undefined or undifferentiable).
  - **Note:** Not all places on the graph where f'(c) = 0 or f'(c) does not exist are extreme points. For example




**Ex #1:** Refer to the sketch of the function below and state whether, at each of the indicated $x$-values, the function has a relative maximum or minimum, a global maximum or minimum, both, or neither.




- Identify all CRITICAL POINTS on the above graph.

**Note: if it is an open interval, the endpoints are not critical points. IF it is a closed interval SOME textbooks consider them to be critical points while some texts do not. Our textbook does NOT.**

- Does the above graph show a closed interval, an open interval or an infinite interval?
- Write the interval using interval notation

**Ex #2:** Find the coordinates of any maximum or minimum values for the function $y = 2x^3 - 24x + 21$.

https://www.desmos.com/calculator/ltkp6rw4w5

- Can you tell if they are maximum values or minimum values from your answer? Can you tell if they are local or absolute extreme values from your answer?

**Ex #3:** Find the critical numbers of the function $f(x) = \frac{2x}{x-3}$

https://www.desmos.com/calculator/dsf1ojnqdd

**Ex #4:** Find the critical numbers for $y = 4x - 6x^{\frac{2}{3}}$

https://www.desmos.com/calculator/sma8m38iop

**THE EXTREME VALUE THEOREM:**
If f(x) is continuous on a **closed interval** [a, b], then f(x) has both a global maximum and a global minimum on this interval.

**Ex #5:** Without drawing a graph, find the absolute extrema of the function $f(x)=x^3 - 12x$ on the interval [-5, 3].

STEPS:
1. Find all critical numbers of f(x) on the given closed interval
2. Find the coordinates of the critical numbers
3. Evaluate the function at the endpoint of the interval to find the endpoint coordinates
4. The coordinate with the largest "y" value in the above steps is the absolute maximum. The coordinate with the lowest y value is the global minimum.

## OUTCOME 5 DAY 1 ASSIGNMENT (Section 4.2 in Text)

**The following questions:**

1. Sketch the graph of a function *that is continuous* on the interval $[-2,3]$ and has:
   (a) a global maximum of 3, a global minimum of 1, and no relative extrema.
   (b) a relative maximum value of 2 at $x=1$, a relative minimum value of 1 at $x=0$ and no other relative extrema; a global minimum of 0 at the right endpoint of the interval and a global maximum of 3 at the left endpoint of the interval.
   (c) a relative and global maximum value of 3 at $x=1$ and a relative and global minimum value of 1 at $x=-1$.
   (d) a critical number at $x=0$ but no relative maximum or minimum value.

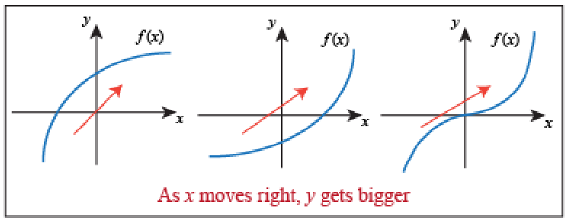
2. Sketch the graph of a function on the interval $[-2,3]$ that has:
   (a) a global maximum but no relative maximum.
   (b) no global maximum and no global minimum.
   (c) a relative maximum and a relative minimum but no global maximum and no global minimum.

**FA The Above Questions Plus: TEXTBOOK P176 #1ii, 2 aefh, 3a-e, 4(without graphing)a-e, g**
**MLA TEXTBOOK P176 #3g-l, 4hi, 6, 7**

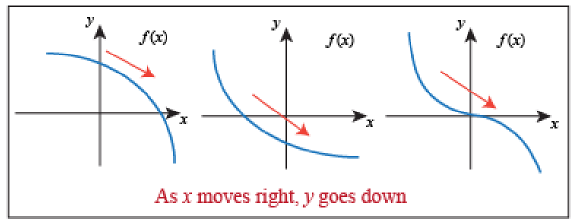
**To determine intervals in which a graph is increasing and decreasing, and to use the First Derivative Test to find these intervals.**

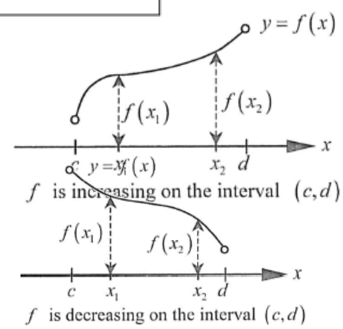
- An intuitive explanation of increasing vs decreasing functions can be found in the images below:

**INCREASING FUNCTIONS**




**DECREASING FUNCTIONS**




- We specifically will talking about INTERVALS where functions increase or decrease.
  - A function, f(x), is said to be increasing on the open interval (c, d) if all $f(x_1) < f(x_2)$ when the order of our x terms satisfies $c < x_1 < x_2 < d$.
  - A function, f(x), is said to be decreasing on the open interval (c, d) if all $f(x_1) > f(x_2)$ when the order of our x terms satisfies $c < x_1 < x_2 < d$

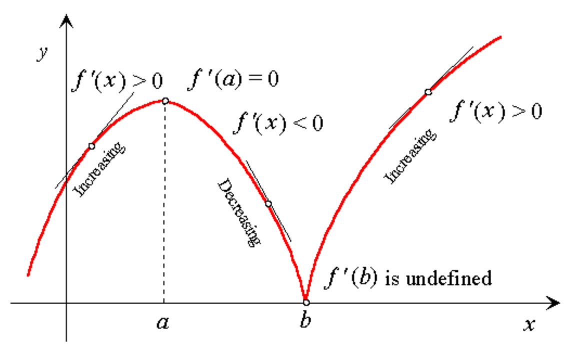
**NOTE: Many University textbooks require that interval of increase and decrease by closed intervals.**

- In general, we say that the derivative of a function will be positive for all values of x in intervals where the graph is increasing and the derivative of a function will be negative for all values of x in intervals where the graph is decreasing.

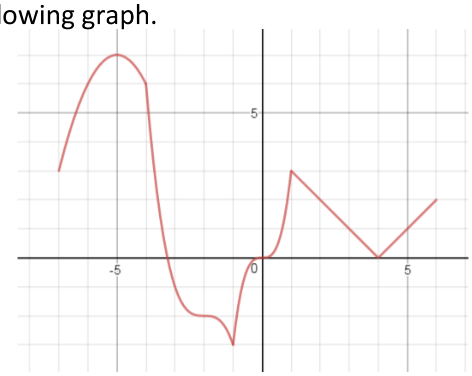
**INCREASING/DECREASING INTERVAL TEST**

Suppose f(x) is a function that is continuous on the open interval (c, d)

- If f'(x) > 0 for all $x \in (c,d)$, with the exception of a finite number of points at which f'(x) = 0 or f'(x) DNA, then f(x) is INCREASING on (c, d)
- If f'(x) < 0 for all $x \in (c,d)$, with the exception of a finite number of points at which f'(x) = 0 or f'(x) DNA, then f(x) is DECREASING on (c, d)




**Ex #1:** State intervals of increase and intervals of decrease for the following graph.



- In the last section we learned to use the first derivative to determine locations of local extrema, but we weren't able to determine if those extrema were maximums or minimums. Today we will learn how to distinguish between maximum and minimum extrema.

**THE FIRST DERIVATIVE TEST:**

If *c* is a critical number of a continuous function f(x), then:

- f(x) has a **Relative Minimum** at the value *x = c* if the value of f'(x) switches signs from negative to positive at *c*
- f(x) has a **Relative Maximum** at the value *x = c* if the value of f'(x) switches signs from positive to negative at *c*

**Ex #2 :** Use sign analysis to find the open intervals in which the function $f(x)=\frac{1}{3}x^3+x^2-3x+1$ is increasing and decreasing. Find the coordinates of any relative extrema and state if they are a maximum or a minimum.

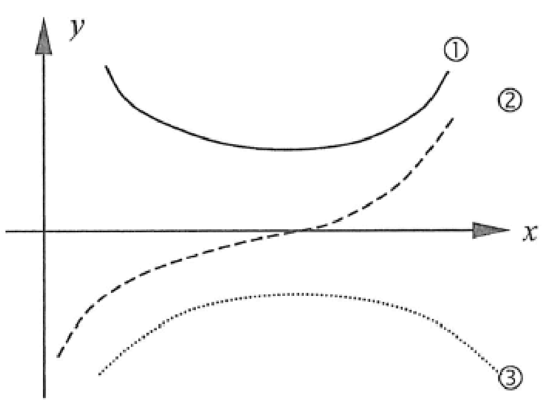
https://www.desmos.com/calculator/yxs0ydzcyg1

**Ex #3 :** Use sign analysis to find the open intervals in which the function $f(x)=3x^{\frac{2}{3}}-6x^{\frac{1}{3}}$ is increasing and decreasing. Find the coordinates of any relative extrema and state if they are a maximum or a minimum.

https://www.desmos.com/calculator/flbq2izeou

**Ex #4 :** Use sign analysis to find the open intervals in which the function $f(x)=\dfrac{x^2}{x-2}, x \neq 2$ is increasing and decreasing. Find the coordinates of any relative extrema and state if they are a maximum or a minimum.

**Ex #5 :** Shown below are the graphs of f(x), f'(x) and another function g(x). Which is which?

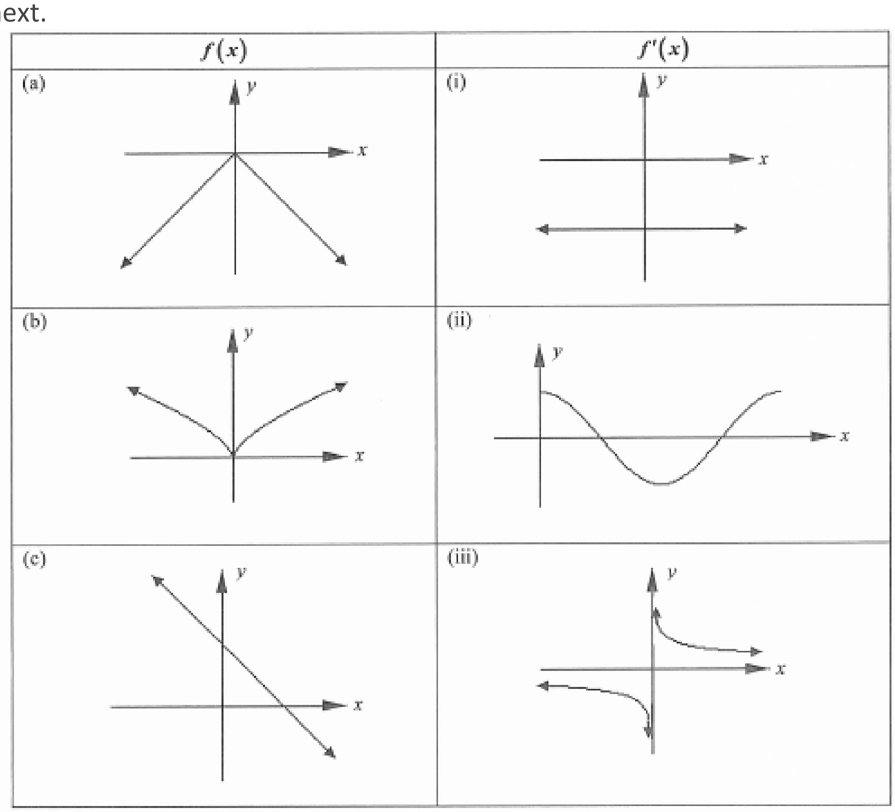


## OUTCOME 5 DAY 2 ASSIGNMENT (Section 4.1 & 4.3 in Text)

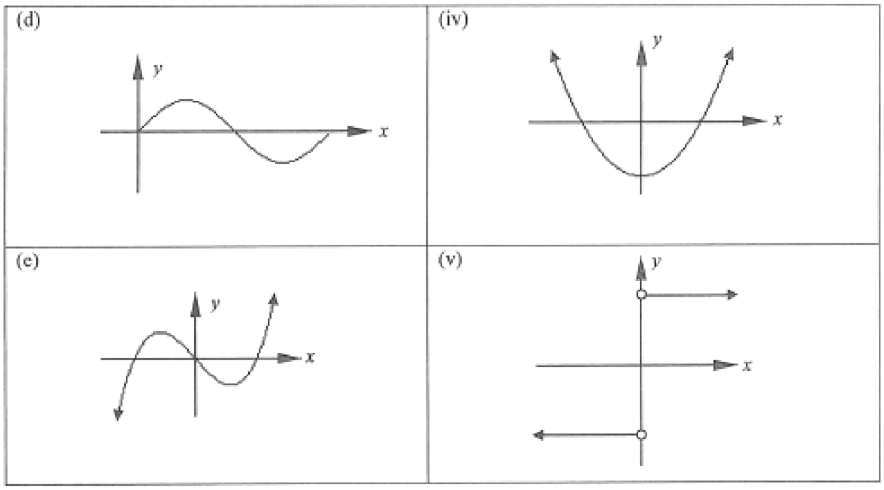
**FA P 170 #1 and the following:**
**Use sign analysis to find the open intervals in which the following functions are increasing and decreasing. Find the coordinates of any relative extrema and state if they are a maximum or a minimum.**

1. $f(x) = 4$
2. $f(x) = 6 - 3x$
3. $f(x) = x^2 + 6x - 8$
4. $f(x) = 8x - 2x^2$
5. $f(x) = x^3 - 27x$
6. $f(x) = x^3 - 3x^2$
7. $f(x) = -x^3 - 3x^2 + 24x + 20$
8. $f(x) = x^4 - 8x^2$
9. $f(x) = 3x^4 + 4x^3 - 36x^2 + 11$
10. $f(x) = 3x^5 - 5x^3$
11. $f(x) = 2\sqrt{x}$
12. $f(x) = 5\sqrt[5]{x}$
13. $f(x) = x - 12x^{1/3}$
14. $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^{2/3}$
15. $f(x) = \frac{2x}{x+2}$
16. $f(x) = \frac{x^2}{x+1}$
17. $f(x) = \sqrt{x}(6 - x)$
18. $f(x) = \frac{x}{\sqrt{x-3}}$

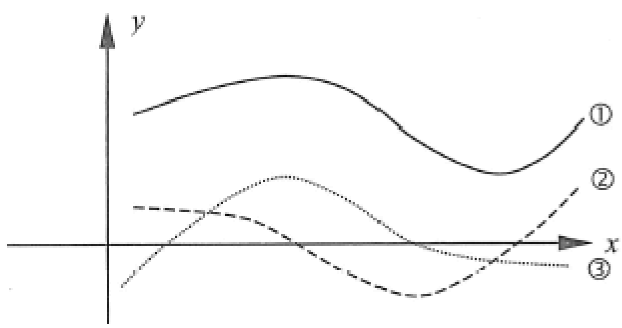
20. Match each function with the graph of its derivative. The scales are not necessarily the same from one graph to the next.




THIS QUESTION IS CONTINUED ON THE NEXT PAGE….. ☺

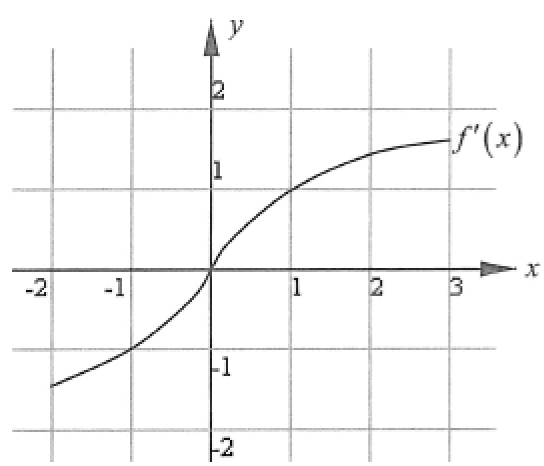



21. Shown below are the graphs of $f(x)$, $f'(x)$, and $g(x)$ (which is not $f'(x)$). Which is which? Explain your reasoning.




**MLA: P 182 #4, 5, 6 & the following**

19. Shown at right is the graph of $f'(x)$ on the interval $[-2,3]$. Answer the following questions about $f(x)$. Explain your reasoning.
   (a) State the open interval(s) on which $f(x)$ is increasing.
   (b) State the open interval(s) on which $f(x)$ is decreasing.
   (c) State the $x$-value(s), if any, at which $f(x)$ has a relative maximum or minimum point.
   (d) Where does $f(x)$ reach its maximum value on the interval $[-2,0]$?

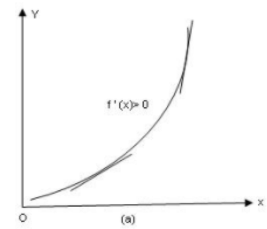


**To determine intervals in which a curve is concave up or down and to find points of inflection.**

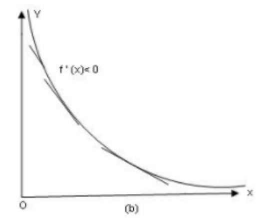
- In the last lesson, we learned that the **first derivative test** tells us where a function f(x) is **increasing** $f'(x) > 0$ and **decreasing** $f'(x) < 0$.
- In this lesson, we will determine what the second derivative tells us about f(x).

**REVIEW: Increasing vs Decreasing**

- A function f(x) is INCREASING on an interval if all slopes of tangent lines are positive f'(x) > 0
    - This graph is going "uphill"

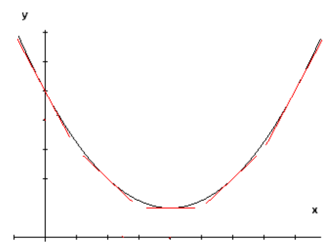



- A function f(x) is DECREASING on an interval if all slopes of tangent lines are negative  f'(x) < 0
    - This graph is going "downhill"




**REVIEW: CONCAVE UP**

- Concave up intervals are shaped like a valley
- Some tangent lines have positive slopes, some negative and one is zero

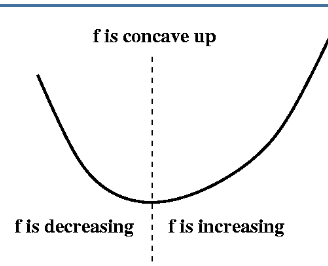


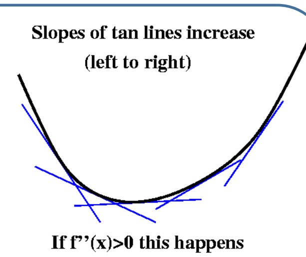

As you move from left to right, are the slopes becoming larger and increasing (more positive) or smaller and decreasing (more  negative)? ______________________

- Your answer to the above question is a way of describing how the slopes of the tangent line are actually changing from left to right – or describing how the first derivative f'(x) is changing.  If you are describing the change of f'(x), you are actually now describing f''(x).  The derivative of a function is a new function that describes the rate of change of the previous function.

**An interval on a function f(x) is CONCAVE UP if**

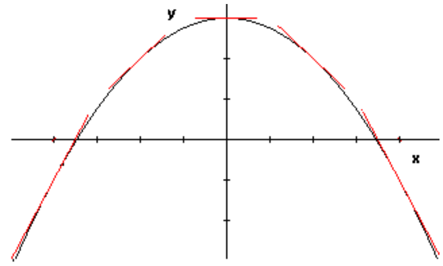
- The first derivative is always increasing (if the slopes of the tangent lines are moving from small to big).
- ∴ Concave up intervals are where f''(x) >0



## REVIEW: CONCAVE DOWN

- Concave down intervals are shaped like a hill
- Some tangent lines have negative slopes, some positive and one is zero

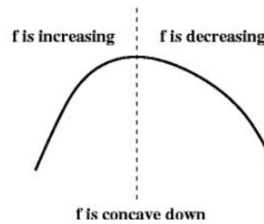


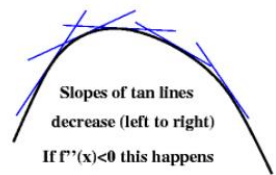

As you move from left to right, are the slopes becoming larger and increasing (more positive) or smaller and decreasing (more negative)? ______________________

- Your answer to the above question is a way of describing how the slopes of the tangent line are actually changing from left to right – or describing how the first derivative f'(x) is changing. If you are describing the change of f'(x), you are actually now describing f''(x). The derivative of a function is a new function that describes the rate of change of the previous function.

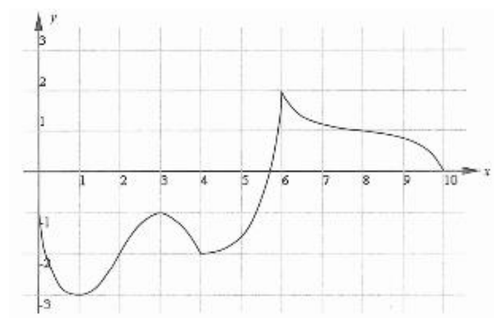
**An interval on a function f(x) is CONCAVE DOWN if**

- The first derivative is always decreasing (if the slopes of the tangent lines are moving from big to small).
- ∴ Concave down intervals are where f"(x) < 0



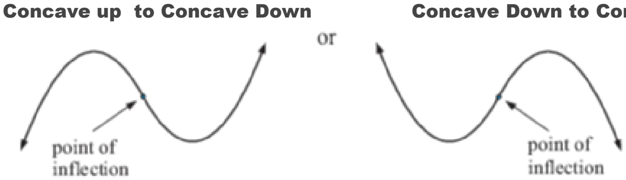

**Ex #1:** State the open intervals on which the function is concave up and concave down.

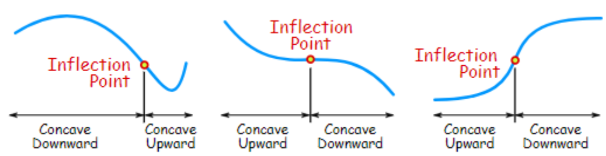



## POINT OF INFLECTION:

- Points where the concavity of a function changes

**Concave up to Concave Down** or **Concave Down to Concave Up**




- At the point of inflection, $f''(x) = 0$ or $f''(x)$ does not exist
  - WORD OF WARNING: In order for the value of x where f"(x) = 0 or f"(x) does not exist to be called a Point of Inflection, the point must be included as a point in the original domain of f(x) and the concavity must be different on either side of the point(the concavity must change) You need to do a sign analysis on f"(x) to verify that the concavity changes!

**Ex #2:** Find the open intervals on which $f(x) = x^3 - 6x^2 - 2x + 4$ is concave up and concave down. Use $f''(x)$ and sign analysis.

- Do a quick sketch of f(x) using the intervals of concavity above.
- Can you identify the point where the graph changes concavity?

**Ex #3:** Given $f(x) = \dfrac{24}{x^2 + 12}$, find the points of inflection and the open intervals where it is concave up and down.

**Ex #4:** Given $f(x)=3x^{\frac{2}{3}}-6x^{\frac{1}{3}}$ find the points of inflection and the open intervals where it is concave up and down. https://www.desmos.com/calculator/80u973xed1

## OUTCOME 5 DAY 3 ASSIGNMENT (Section 5.3 in Text) :

**Text P229 #1, 2, 3abcd**

**To apply the first and second derivatives to determine relative and absolute extrema.**

**The second derivative:** In some situations, we can use both the first and second derivative to determine local maximum and minimum values of a function f(x).

Look at the graphs of f(x) on the right. What do we know about f'(c) and f"(c)?

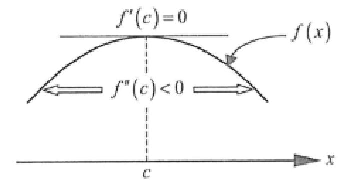


Figure 8

In *Figure 8*, when x = c:

- The slope of the tangent line is 0, therefore **f'(c) = 0**. This indicates that **f(c)** is a **relative max or min** value.
- The slope of the tangent lines are decreasing, therefore **f"(c) < 0**. This indicates that the curve is **concave down**, which means that **f(c)** must be a **relative maximum** value.

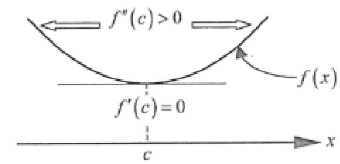


Figure 9

In *Figure 9*, when x = c:

- The slope of the tangent line is 0, therefore **f'(c) = 0**. This indicates that **f(c)** is a **relative max or min** value.
- The slope of the tangent lines are increasing, therefore **f"(c) > 0**. This indicates that the curve is **concave up**, which means that **f(c)** must be a **relative minimum** value.

**THE SECOND DERIVATIVE TEST:**

Suppose that f(x) is a continuous function in an interval containing x = c.

- If f'(c) = 0 **and** f''(c)>0, then f has a local minimum at c.
- If f'(c) = 0 **and** f''(c)<0, then f has a local maximum at c.
- CAUTION: If f"(c) = 0 or f"(c) DNE, the second derivative test fails. You can then use the first derivative test (which never fails) to locate a relative extrema. You would then need to do a sign analysis to determine whether or not f(c) is a relative maximum or minimum or neither.

**Ex #1:** Use the second derivative test to find the local max and min values of $f(x)=x^3-12x+5$.

**Ex #2:** Use the second derivative test to find the relative extrema of $f(x)=\frac{1}{4}x^4-x^3$.

WORD OF WARNING: : If f”(c) = 0 or f”(c) DNE, the second derivative test fails. You can then use the first derivative test (which never fails) to locate a relative extrema. You would then need to do a sign analysis to determine whether or not f(c) is a relative maximum or minimum or neither.

**Ex #3 :** By examining the graph of f(x), determine if f(x), f’(x) and f”(x) is positive, negative or zero at the indicated points. Assume that points that appear to be relative extrema or inflection points are so.

*(Tip: find the zero values first.)*

| | A | B | C | D | E | F | G | H | I | J | K | L | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| f(x) | – | | | | | | | | | | | | |
| f’(x) | + | | | | | | | | | | | | |
| f”(x) | – | | | | | | | | | | | | |

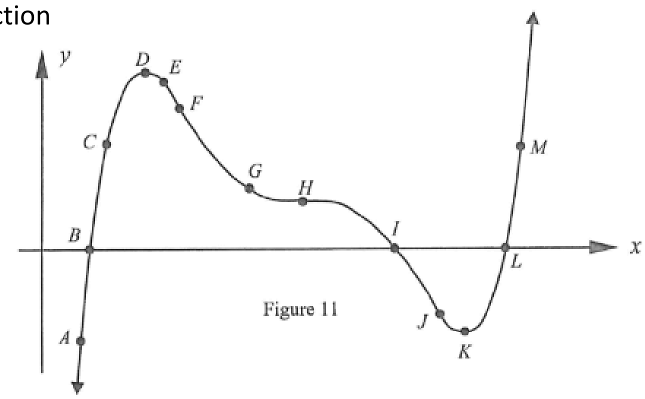


Figure 11

## OUTCOME 5 DAY 4 ASSIGNMENT (Section 5.4 in Text) :

**TEXT P232 #1 even, 2bef and the following:**

1. Sketch the graph of a function $f(x)$, in the vicinity of $x=2$, for which:

(a) $f(2)>0$, $f'(2)>0$, $f''(2)>0$ (b) $f(2)>0$, $f'(2)>0$, $f''(2)<0$

(c) $f(2)>0$, $f'(2)<0$, $f''(2)>0$ (d) $f(2)>0$, $f'(2)<0$, $f''(2)<0$

(e) $f(2)>0$, $f'(2)=0$, $f''(2)>0$ (f) $f(2)>0$, $f'(2)=0$, $f''(2)<0$

(g) $f(2)>0$, $f'(2)>0$, $f''(2)=0$ (h) $f(2)>0$, $f'(2)<0$, $f''(2)=0$

(i) $f(2)>0$, $f'(2)=0$, $f''(2)=0$

# CALCULUS 30: OUTCOME 5 DAY 5 – VERTICAL ASYMPTOTES (TEXTBOOK: 5.1)

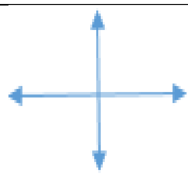
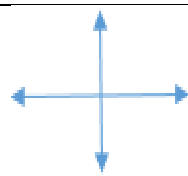
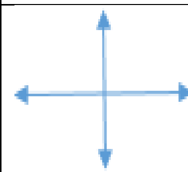
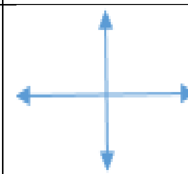
**To apply understanding of limits to determine equations of vertical asymptotes.**

We will soon learn how to sketch the graph of a function, but before doing so, we need to review the following:

## Vertical Asymptote (VA) - REVIEW

- As discussed in Outcome 3, a vertical asymptote (VA) occurs in a rational function at values of x that give $\frac{\#}{0}$.
- To find the vertical asymptote, you will need to factor the denominator *(and possibly the numerator).*

**Examples:**

| | $f(x)=\frac{x}{x-2}$ | $f(x)=\frac{-3}{x^2-1}$ | $f(x)=\frac{4x}{x^2+4}$ | $f(x)=\frac{x-3}{x^2-9}$ |
|---|---|---|---|---|
| VA at x = ? | | | | |
| Sketch the asymptote |  |  |  |  |

Can we justify what we learned about vertical asymptotes in PC 30 using limits?

**VERTICAL ASYMPTOTE(S):**
The line x = a is a VERTICAL ASYMPTOTE of the graph of a function y = f(x) if either of the following is true

$\lim_{x\to a^+} f(x)=\pm\infty$ or $\lim_{x\to a^-} f(x)=\pm\infty$

1. Note that it only takes one of these to be proven true in order for x = c to be an asymptote, but both may be true
2. Beyond the constraints of the functions studied in PC30, there are functions where asymptotes don't occur at places where the denominator is zero. For example the function $y=\frac{\sin x}{x}$ does NOT have an asymptote at x = 0

https://www.desmos.com/calculator/erjzgs6iq8

**Ex1:** Find $\lim_{x\to 6}\left(2-\frac{5}{(x-6)^2}\right)$ and determine if there is a vertical asymptote at x = 6

$\lim_{x\to 6^-}\left(2-\frac{5}{(x-6)^2}\right)$

$\lim_{x\to 6^+}\left(2-\frac{5}{(x-6)^2}\right)$

**Ex2:** Consider the rational function $f(x)=\frac{x^2}{\sqrt{x+1}}$.

a) Determine the equations of the vertical asymptotes using limits.
b) Determine the behavior of the function on each side of these VAs and sketch the graph near the asymptotes.

**Ex3:** Consider the rational function $f(x) = \dfrac{3x-9}{x^2 - x - 2}$.

a) Determine the equations of the vertical asymptotes using limits.
b) Determine the behavior of the function on each side of these VAs and sketch the graph near the asymptotes.

## OUTCOME 5 DAY 5 ASSIGNMENT (Section 5.1 in Text) :

**FA P212 #1, 2bdf-l, 3ace-h**

# CALCULUS 30: OUTCOME 5 DAY 6 – HORIZONTAL & SLANT ASYMPTOTES (TEXTBOOK: 5.2 & 5.6)

**To review how to find x and y intercepts. To apply understanding of limits to determine equations of horizontal asymptotes. To learn how to find equations of oblique/slant asymptotes..**

## Finding Intercepts:

1. To find y-intercepts, we let x = 0 and solve for y or f(0).
2. To find x-intercepts, we let y = 0 or f(x) = 0 and solve for x.
   When working if rational functions, we would solve for numerator = 0. Don't forget to verify your answer by substituting it into the original function.

**Ex #1:** Find the intercepts of $f(x)=\dfrac{3x-9}{x^2-x-2}$.

**HORIZONTAL ASYMPTOTE(S):**

The line y = b is a HORIZONTAL ASYMPTOTE of the graph of a function y = f(x) if either of the following is true

$\lim\limits_{x\to\infty} f(x)=b$ or $\lim\limits_{x\to-\infty} f(x)=b$

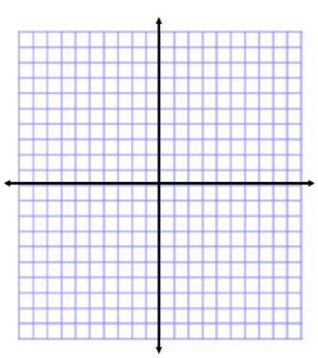
3. Note that a function can have more than one horizontal asymptote
   - We learned how to take limits at infinity in our first unit on the day titled "Outcome 3 Day 3 – Limits at Infinity". The video links for that section were a) https://goo.gl/kY6D4t b) https://goo.gl/5RLxzV

**Ex #2:** Find the horizontal asymptotes of the following functions.

a) $f(x)=\dfrac{3x-9}{x^2-x-2}$

b) $f(x)=\dfrac{6x^3+1}{x^3-4x^2}$

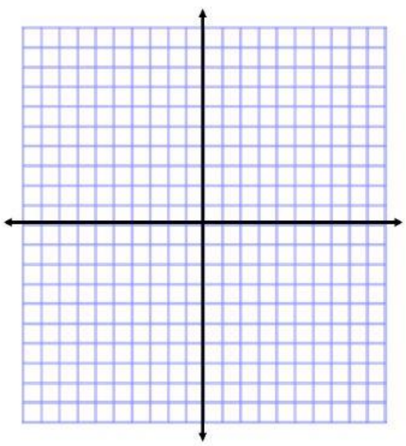
c) Find the intercepts of question a) above and sketch using the intercepts and the HA.



- Functions that contain radical signs or absolute value signs may have two asymptotes, one as $x \to \infty$ and a different one as $x \to -\infty$

**Ex #3:** Use limits to find the two horizontal asymptotes and the vertical asymptote for the function $f(x) = \dfrac{\sqrt{x^2+4}}{x-2}$.

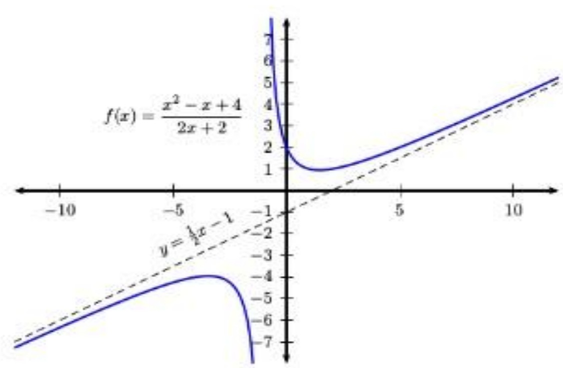
Use them to sketch the function.



https://www.desmos.com/calculator/fphk0m01fb

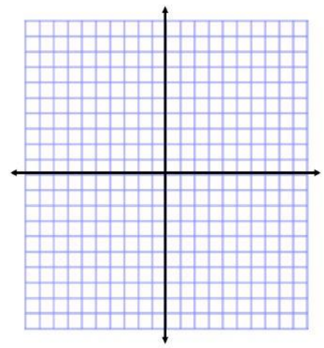
**OBLIQUE/SLANT ASYMPTOTES** : If the degree of the numerator of a rational function f(x) is one larger than the degree of the denominator, the rational function as an **oblique/slant** asymptote line. (SA or OA)

**To find the equation of the asymptote of f(x)**




- Perform long or synthetic division on f(x) (You may use synthetic division if the denominator of f(x) is a binomial of degree one)
- Write out the answer in the form $y = mx + b + \dfrac{\text{remainder}}{\text{denominator of f(x)}}$
- Consider what happens to the function as $x \to \infty$. The Y = mx + b part of the equation will continue to grow as $x \to \infty$ but the remainder/denominator portion will tend to zero as $x \to \infty$
- Because the remainder will tend to zero, the equation of the asymptote as $x \to \infty$ will be just y = mx + b

**Ex #2:** Use synthetic division to find the equation of the oblique asymptote to the function $f(x)=\dfrac{x^2-x-7}{x-3}$.

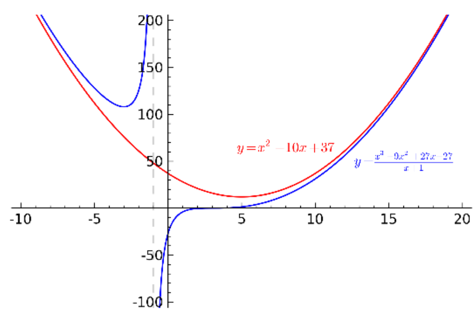


Are there any other asymptotes?

Can you draw a quick sketch of the graph?

**Ex #2:** Use long division to find the oblique/slant asymptote of the curve $y=\dfrac{2x^3-3x^2+x-3}{x^2+1}$

**Parabolic Asymptote**: If the degree of the numerator of a rational function is two larger than the degree of the denominator, the rational function has a **parabolic** asymptote line. (We will not be working a lot with these).




- You may have noticed that you can determine the HA by examining the degrees of both the numerator and denominator:

Let n = degree of numerator and
d = degree of denominator

a) If n < d, there is a horizontal asymptote at y = 0.

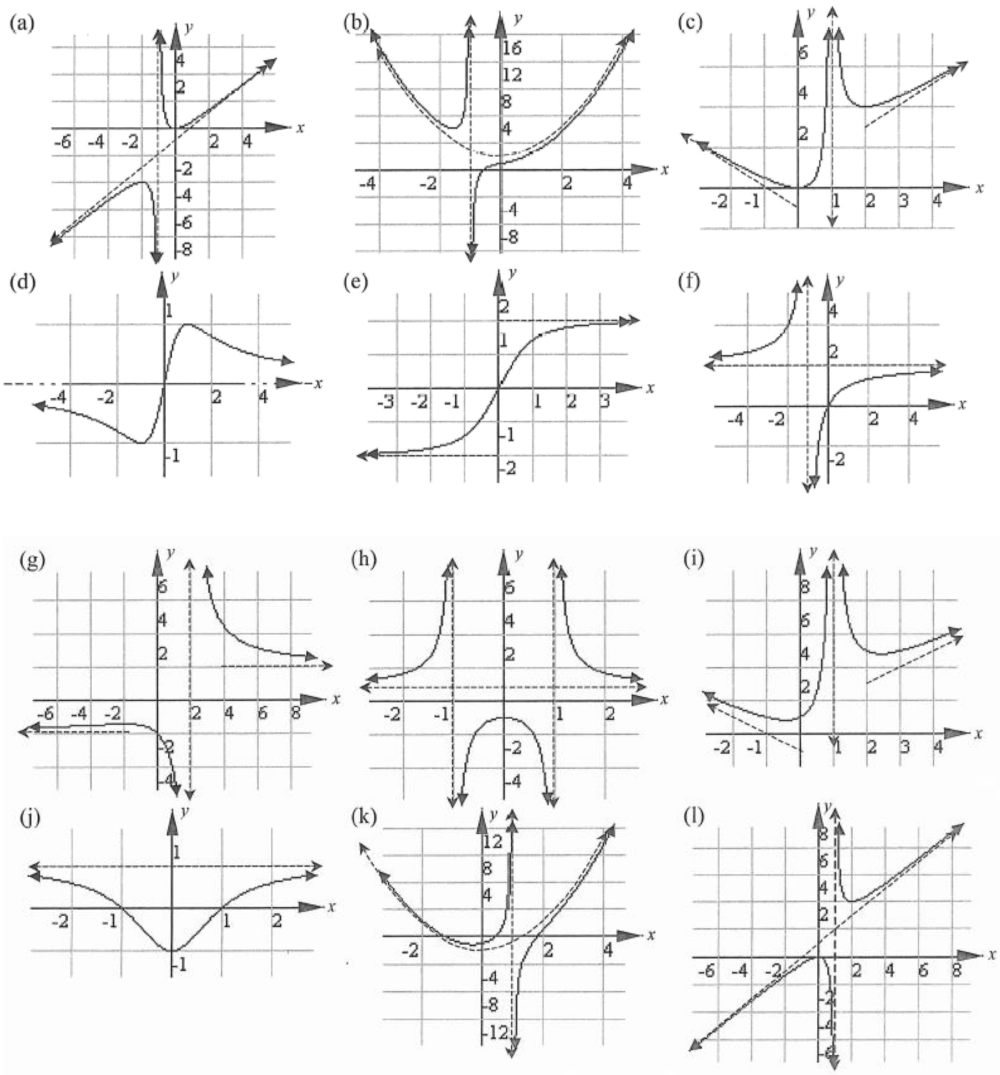
b) If n = d, there is a horizontal asymptote at $y=\dfrac{a}{b}$.

c) If n > d, there is a slanted or parabolic asymptote.

## OUTCOME 5 DAY 6 ASSIGNMENT (Section 5.2 & 5.6 in Text) :

**FA P222 #1, 3abc, 4** **P244 #1ace** **PLUS match the following**

| | | | |
|---|---|---|---|
| 1. $f(x)=\dfrac{2x}{x+1}$ | 2. $f(x)=\dfrac{2x}{x^2+1}$ | 3. $f(x)=\dfrac{x^2}{x-1}$ | 4. $f(x)=\dfrac{x^2}{x+1}$ |
| 5. $f(x)=\dfrac{x^2-1}{x^2+1}$ | 6. $f(x)=\dfrac{x^2+1}{x^2-1}$ | 7. $f(x)=\dfrac{x^3+x^2+2x+1}{x+1}$ | 8. $f(x)=\dfrac{x^3-x^2-2x+1}{x-1}$ |
| 9. $f(x)=\dfrac{2\sqrt{x^2+4}}{x-2}$ | 10. $f(x)=\dfrac{2x}{\sqrt{x^2+1}}$ | 11. $f(x)=\dfrac{x^2}{\lvert x-1\rvert}$ | 12. $f(x)=\dfrac{x^2+1}{\lvert x-1\rvert}$ |



To combine all the concepts in this outcome in order to sketch the graph of a function f(x).

**STEPS TO CURVE SKETCHING:**

1. Find the Domain and any restrictions
2. Find f'(x).
    a) Find the critical numbers by finding where f'(x) = 0 and where f'(x) does not exist
    b) Use f'(x) and the critical numbers to do a sign analysis to find the open intervals where f(x) is increasing and/or decreasing
    c) Use the critical numbers (and any endpoints) to find the coordinates of the relative extrema
3. Find f"(x).
    a) Find possible inflection points by finding where f"(x) = 0 and f"(x) does not exist.
    b) Use f"(x) and the possible inflection points and do a sign analysis. Use this find the open interval on which f(x) is concave up and concave down, and to verify if all possible points of inflection are actually points of inflection (they are only points of inflection if they occur at points where the concavity of f(x) changes)
4. Find the x and y intercepts
5. Find the vertical asymptote(s) using limits (this helps us determine if the graph is moving up towards $+\infty$ along the asymptote or down towards $-\infty$)
6. Find the horizontal asymptote(s) using limits as $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$.
7. **USE ALL OF THE ABOVE INFORMATION TO SKETCH THE GRAPH ☺**

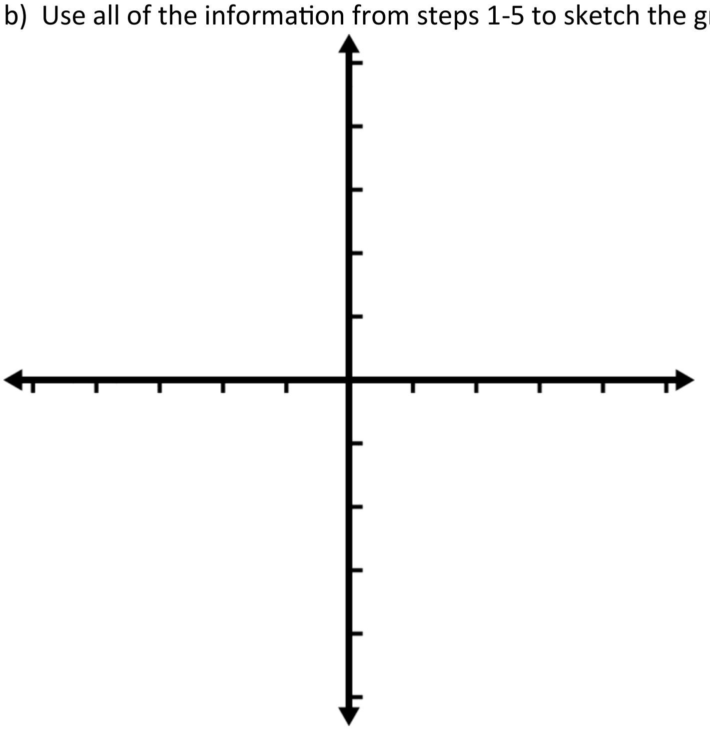
**NOTE: Notes are on the following page (each example needs a full page). The assignment for the end of this section is as follows:**

## OUTCOME 5 DAY 7 ASSIGNMENT (Section 5.5 in Text) :

**FA P240 #1-12 (Use the steps given in the notes above. The textbook question asks for similar steps but also asks for symmetry (which isn't necessary to find))**

**Ex #1:** a) Do steps 1-5 from above for the function $y = x^4 + 4x^3$

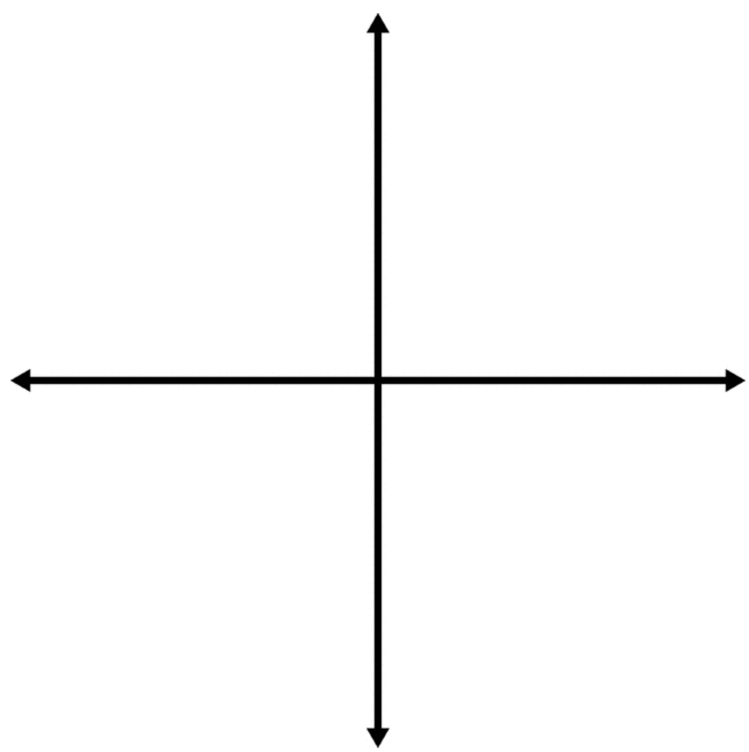
b) Use all of the information from steps 1-5 to sketch the graph



**Suggestions for Making your Sketch:**

- Pick your scale. To choose your scale, look at the coordinates of specific points you have found and the equations of asymptotes. Often the vertical scale will jump by larger amounts than the horizontal scale. It may take some adjustment to get the scale correct
- Draw any asymptotes
- Mark the intercepts
- Plot the relative extrema. At each relative maximum, make a small hill $\cap$ ; at each relative minimum, make a small valley $\cup$
- Plot the points of inflection
- Use the increasing/decreasing intervals and the concavity intervals together with the asymptotes to complete the graph
- Occasionally you may wish to plot one or two additional points in regions that need further clarification

**Ex #2:** Use all of steps 1-6 to sketch the function $y = x^{\frac{2}{3}} + \frac{1}{5}x^{\frac{5}{3}}$

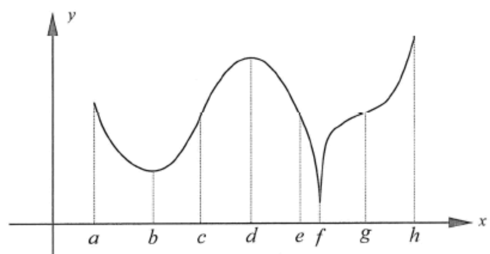


**(Day 2) Ex #3:** Use all of steps 1-6 to sketch the function $f(x) = \dfrac{x^2}{1-x^2}$

# OUTCOME 5 REVIEW

**P 196 #1, 2, 3, 4**
**P 245 # 2(using limits), 3abc, 5 (use all 6 steps from our notes)**
**ADDITIONAL QUESTIONS:**

1. Refer to the graph of the function shown below right. For each of the $x$-values $a$, $b$, $c$, $d$, $e$, $f$, $g$, and $h$, choose the words "absolute maximum", "absolute minimum", "relative maximum", "relative minimum", or "none of these". For some $x$-values, it may be necessary to use more than one set of words.




Question 3

2. Sketch the graph of a function that is continuous on $[2,5]$ and has all of the following properties:
   - absolute maximum at $x=2$
   - absolute minimum at $x=5$
   - relative minimum at $x=3$
   - relative maximum at $x=4$

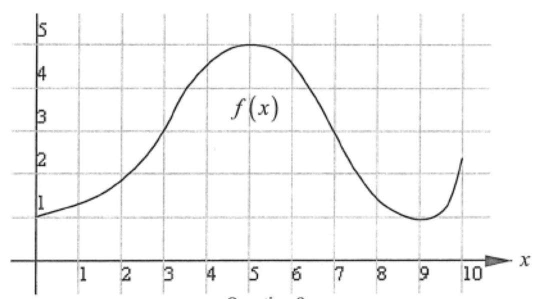
3. Sketch the graph of a function that is continuous on $[2,5]$ and has all of the following properties:
   - absolute minimum at $x=2$
   - absolute maximum at $x=5$
   - relative minima at $x=3.5$ and $x=4.5$
   - relative maxima at $x=3$ and $x=4$

4. Sketch the graph of a continuous function that has a relative maximum at $x=2$ and is differentiable at $x=2$.

5. Sketch the graph of a continuous function that has a relative maximum at $x=2$ but is not differentiable at $x=2$.

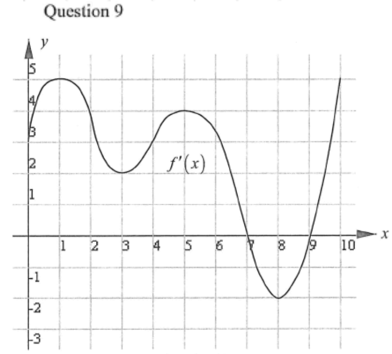
6. Sketch the graph of a function on the interval $[2,5]$ that has an absolute maximum at $x=4$ but is not continuous at $x=4$.

7. Use the graph of $f$ at right to find:
   (a) the open interval(s) on which $f$ is increasing.
   (b) the open interval(s) on which $f$ is decreasing.
   (c) the open interval(s) on which $f$ is concave up.
   (d) the open interval(s) on which $f$ is concave down.
   (e) the coordinates of any relative extrema.
   (f) the coordinates of any inflection points.




Question 9

8. The graph of **the derivative of a function $f$** is shown below right. Use the graph to answer the following question about $f$.
   (a) On what open interval(s) is $f$ increasing?
   (b) On what open interval(s) is $f$ decreasing?
   (c) On what open interval(s) is $f$ concave down?
   (d) On what open interval(s) is $f$ concave up?
   (e) State the $x$-value(s) at which $f$ has a point of inflection.
   (f) State the $x$-value(s) at which $f$ has any relative extrema.
   (g) What are the critical numbers?

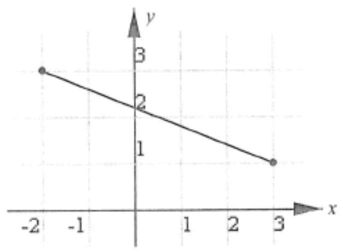


## SOLUTIONS TO: OUTCOME 5 DAY 1 ASSIGNMENT

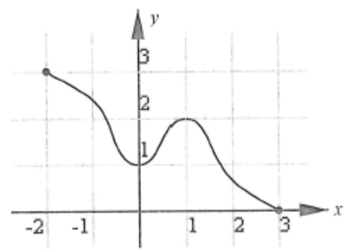
**EXTRA QUESTIONS:**

There are many possible answers for questions 1 and 2.

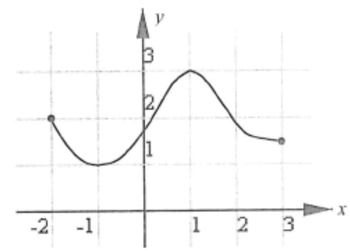
1. (a)



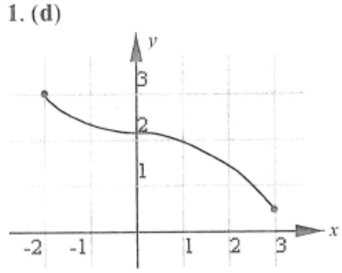

(b)



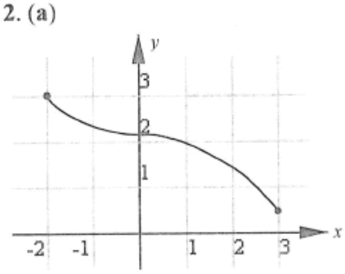

(c)



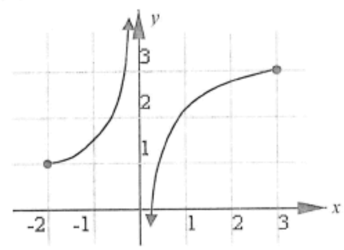

1. (d)



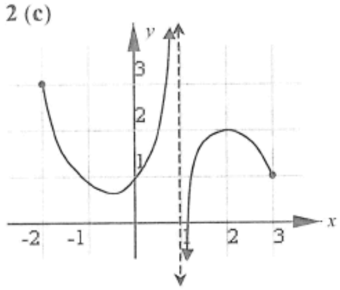

2. (a)



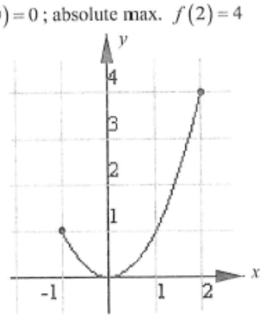

(b)



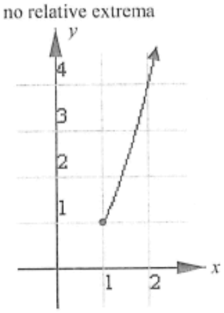

2 (c)




3. (a) relative and absolute min. $f(0)=0$; absolute max. $f(2)=4$




(b) absolute min. $f(1)=1$; no absolute max.; no relative extrema



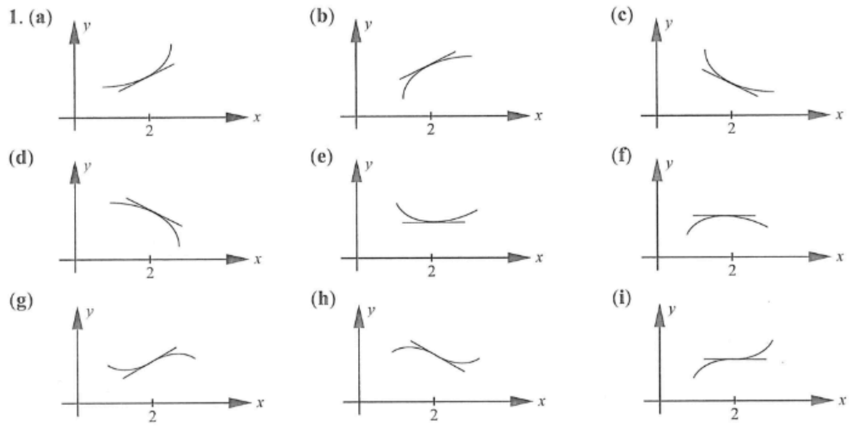
## SOLUTIONS TO: OUTCOME 5 DAY 2 ASSIGNMENT

**1.** never increasing or decreasing; no relative extrema **2.** never increasing; decreasing for $x \in (-\infty,\infty)$; no relative extrema **3.** decreasing for $x \in (-\infty,-3)$; increasing for $x \in (-3,\infty)$; relative minimum at $(-3,-17)$ **4.** decreasing for $x \in (2,\infty)$; increasing for $x \in (-\infty,2)$; relative maximum at $(2,8)$ **5.** decreasing for $x \in (-3,3)$; increasing for $x \in (-\infty,-3) \cup (3,\infty)$; relative maximum at $(-3,54)$; relative minimum at $(3,-54)$ **6.** decreasing for $x \in (0,2)$; increasing for $x \in (-\infty,0) \cup (2,\infty)$; relative maximum at $(0,0)$, relative minimum at $(2,-4)$ **7.** decreasing for $x \in (-\infty,-4) \cup (2,\infty)$; increasing for $x \in (-4,2)$; relative minimum at $(-4,-60)$; relative maximum at $(2,48)$ **8.** decreasing for $x \in (-\infty,-2) \cup (0,2)$; increasing for $x \in (-2,0) \cup (2,\infty)$; relative maximum at $(0,0)$; relative minima at $(-2,-16)$ and $(2,-16)$
**9.** decreasing for $x \in (-\infty,-3) \cup (0,2)$; increasing for $x \in (-3,0) \cup (2,\infty)$; relative maximum at $(0,11)$; relative minima at $(-3,-178)$ $(2,-53)$ **10.** decreasing for $x \in (-1,1)$; increasing for $x \in (-\infty,-1) \cup (1,\infty)$; relative maximum at $(-1,2)$; relative minimum at $(1,-2)$ **11.** never decreasing; increasing for $x \in (0,\infty)$; no relative extrema **12.** never decreasing; increasing for $x \in (-\infty,\infty)$; no relative extrema **13.** increasing for $x \in (-\infty,-8) \cup (8,\infty)$; decreasing for $x \in (-8,8)$; relative maximum at $(-8,16)$; relative minimum at $(8,-16)$ **14.** increasing for $x \in (-\infty,-1) \cup (0,\infty)$; decreasing for $x \in (-1,0)$; relative maximum at $(-1,7/6)$; relative minimum at $(0,0)$ **15.** never decreasing; increasing for $x \in (-\infty,-2) \cup (2,\infty)$; no relative extrema

**16.** decreasing for $x \in (-2,-1) \cup (-1,0)$; increasing for $x \in (-\infty,-2) \cup (0,\infty)$; relative maximum at $(-2,-4)$, relative minimum at $(0,0)$ **17.** increasing for $x \in (0,2)$; decreasing for $x \in (2,\infty)$; relative maximum at $\left(2,4\sqrt{2}\right)$ **18.** decreasing for $x \in (3,6)$; increasing for $x \in (6,\infty)$; relative minimum at $\left(6,2\sqrt{3}\right)$ **19.** **(a)** increasing for $x \in (0,3)$ since $f'(x) > 0$ **(b)** decreasing for $x \in (-2,0)$ since $f'(x) < 0$ **(c)** there is a relative minimum at $x = 0$ because $f'(x)$ switches from negative, to 0, to positive there; there is no relative maximum **(d)** Since $f(x)$ decreases on the interval $(-2,0)$ because $f'(x) < 0$, $f(x)$ must reach its maximum value at $x = -2$ **20.** **(a)** v **(b)** iii **(c)** i **(d)** ii **(e)** iv **21.** ③ could not be the derivative of ① since initially ① is increasing while ③ is negative. ① could not be the derivative of ③ since ① is never negative but in places ③ decreases. Similarly, ① could not be the derivative of ②. ② could not be the derivative of ③ since ② becomes positive (near the right end) while ③ is still decreasing. ③ could not be the derivative of ② because near the left end ② is decreasing where ③ is positive. ② is the derivative of ① since ② is positive where ① increases and ② is negative where ① decreases. ② is zero where ① has a relative maximum or relative minimum. So ① is $f(x)$, ② is $f'(x)$, and ③ is $g(x)$.

## SOLUTIONS TO: OUTCOME 5 DAY 4 ASSIGNMENT

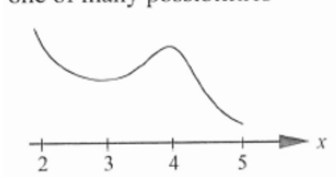
1.



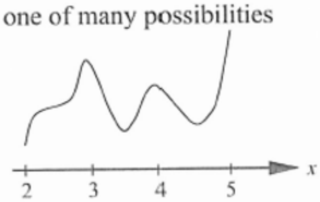

## SOLUTIONS TO: REVIEW

1. a- none, b – relative minimum, c – none of these (it is an inflection point though!) , d – relative and absolute maximum, e – none of these, f – relative and absolute minimum, g – none of these (it is an inflection point though!), h - global maximum

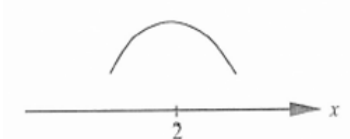
2. one of many possibilities



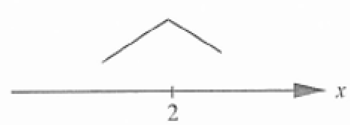
3. one of many possibilities



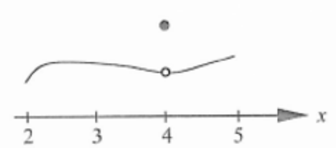
4. one of many possibilities



5. one of many possibilities



6. one of many possibilities



7 (a) $(0,5)\cup(9,10)$ (b) $(5,9)$
(c) $(0,3)\cup(7,10)$ (d) $(3,7)$
(e) $(5,5)$ is a relative maximum and $(9,1)$ is a relative minimum.
(f) $(3,3)$ and $(7,3)$

8 (a) $(0,7)\cup(9,10)$ (b) $(7,9)$ (c) $(1,3)\cup(5,8)$ (d) $(0,1)\cup(3,5)\cup(8,10)$ (e) $x\in\{1,3,5,8\}$
(f) relative maximum at $x=7$; relative minimum at $x=9$ (g) $x\in\{7,9\}$