## AVERAGE VELOCITY/SPEED:

- Average velocity $(x, y)$ and $\left(x_{1}, y_{1}\right)$ on a distance time graph where $y$ is distance and $x$ is time is given by

$$
\text { Average Velocity }=\frac{\text { distance travelled }}{\text { time elapsed }}=\frac{\text { change in distance }}{\text { change in time }}=\frac{\Delta y}{\Delta x}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}
$$

- Often the variables used are not $x$ and $y$. Instead of $y$ they often use "s" ( $s$ is used in place of the word distance - $s$ stands for the Latin word SPATIUM meaning distance). Instead of $x$ they often use " $t$ ". In that case we would say that

$$
\text { Average Velocity }=\frac{\text { change in distance }}{\text { change in time }}=\frac{\Delta s}{\Delta t}=\frac{s_{2}-s_{1}}{t_{2}-t_{1}}
$$

- The distance fallen (d) for an object dropped at rest is given by the following (you learned these in Science 10) $\mathrm{d}(\mathrm{t})=4.9 \mathrm{t}^{2}$ (when d is measured in m and time in s ) $d(t)=16 \mathrm{t}^{2}$ (when d is measured in ft and time in s )
- When an object falls, it is constantly changing speed (it is accelerating). As a result, the instantaneous speed (the speed at a given point in time) will not usually match the average speed. If you try to calculate instantaneous speed using the average speed formula you will have a problem since you will be dividing by zero.
- Isaac Newton developed calculus to get around the problem of dividing by zero. Calculus is a powerful tool where you can solve problems that involve dividing by zero (using limits to find a derivative), and it allows you to add an infinite number of items (using limits to find an integral).

Example 1: A rock breaks loose from the top of a tall cliff. What is its average speed during the first 2 seconds of fall? NOTE: Because we use $x$ and $y$ so often in algebra, let's change the formula above to $y=16 x^{2}$. The variable $y$ will give the distance, while $x$ will represent time. This means that the speed formula becomes the slope formula. Plug in the formula for distance ( $y_{2} \& y_{1}$ ) to find a slope formula in terms of the variable $x$, then solve the problem numerically.


## How would you answer the following questions?

- What is the limit to the number of questions I will assign you daily?
- What is the limit of the number of times I will ask you to put your phone away before I get cranky?
- What is the limit to the speed you can drive on a highway?
- What is the limit to how high a ball will bounce if you let if drop from your hand?
- Most of us know what the word LIMIT means outside of the world of mathematics. Is there a definite answer to each of these questions? What does your answer mean? How can you ensure that your answer is as accurate as possible?


## Instantaneous speed

- is hard to define because it is a speed at time zero. We need to use the concept of limit to define it.
- Instantaneous speed does not usually match average speed.
- You cannot use the average speed formula when you calculate instantaneous speed because it will cause you to divide by zero.

Example 2: Find the Instantaneous speed of the rock in Example 1 at the instant $t=2$.
Now we have a problem. The speed at exactly 2 seconds implies that no time has elapsed (i.e. the change in time is 0 ) and we cannot divide by zero. We will need to introduce the calculus concept of the limit. We need to use our imagination a little bit. First assume that time one is given by $x_{1}=2$. Next, we imagine that a time a little bit past 2 seconds is the second time. Use $x_{2}=2+h$, where $h$ is a very small amount of time added to 2 seconds. " $h$ " is so small that we say it is approaching zero. Let's apply this to the slope formula

Let's look at a chart of values as the value of " $h$ " gets smaller, then find the answer.

| $\frac{\Delta y}{\Delta t}=\frac{16(2+h)^{2}-16(2)^{2}}{h}$ |  |
| :---: | :---: |
| Length of | Average Speed <br> for Interval <br> Time Interval, <br> $h(\mathrm{sec})$ |
| 1 | $8 y / \Delta t(\mathrm{ft} / \mathrm{sec})$ |
| 0.1 | 60 |
| 0.01 | 64.6 |
| 0.001 | 64.016 |
| 0.0001 | 64.0016 |
| 0.00001 | 64.00016 |

Example 3: The following table indicates the number of homework questions that Ms. C gave over the course of a week. What does the limit of the number of questions appear to be? Partial questions indicate questions were only some of parts $a, b, c$, etc. that were assigned.

- What is the limit to the number of questions?
- Fill in the following blanks: The limit to the number of questions assigned
is $\mathrm{n}=$ $\qquad$ as d goes towards $\qquad$

| Day (d) | Number of Questions (n) |
| :--- | :--- |
| 1 | 75.5 |
| 2 | 74.75 |
| 3 | 75.5 |
| 4 | 74.375 |
| 5 | 74.3 |
| 6 | 74.25 |
| 7 | 74.214 |
| 8 | 74.188 |
| 9 | 74.167 |

### 2.1 Day 2 - Definition of Limits

VIDEO LINKS: a) https://goo.gl/3Rchor (a bit long - start at about the 15 min mark)
b) https://goo.gl/9h3Upx
c) https://goo.gl/NgjAtc

- Yesterday we calculated the limit of functions measuring instantaneous velocity. The limit was the value of $x$ when you pushed $y$ to a certain value. The answer to the limit of the instantaneous speed in example 2 was $\qquad$

- If you were to graph the functions from yesterday, what does that limit represent graphically?
https://www.desmos.com/calculator/e9e5dysmns
- Graphically, the limit of a function is
$\qquad$ that a function approaches as you approach a certain value for $\qquad$ .

Ex \#1: Given the function $f(x)=x^{2}$, determine the limit as $x$ approaches 2 .

- Mathematically this would be written as follows: If $\mathrm{f}(\mathrm{x})=\mathrm{x}^{2}$, find $\lim _{x \rightarrow 2} f(x)$
- To find this answer, you must try approaching the indicated x value of 2 from BOTH the left and the right side of the graph. This means you must complete the following two calculations:

$$
\circ \quad \lim _{x \rightarrow 2^{-}} f(x)=\square
$$

(This means to "drive" from the left towards $x=2$ ON the graph and see how high you are on the $y$ axis)

> And
> $\circ \lim _{x \rightarrow 2^{+}} f(x)=\square$
(This means to "drive" from the right towards a value of $x=2$ ON the graph and see how high you are on the $y$ axis

- If both of your answers in the above step were the same, you have found the limit! Therefore $\lim _{x \rightarrow 2} f(x)=\square$

- NOTE: the function does not actually have to exist at the value of the limit - there can be a hole or an asymptote at the actual location! The limit represents the extreme height that you approach as you get infinitely closer as you "drive" from the left and right on the graph!


## INFORMAL DEFINITION OF A LIMIT :

Shown at right is the graph of the function $f(x)=(x-1)^{2}$. By examining the graph, what is $\lim _{x \rightarrow 3}(x-1)^{2}$ ? That is, as input values, $x$, approach 3 from either side, what do the output values, $(x-1)^{2}$, approach? By examining the graph, we see that as $x \rightarrow 3,(x-1)^{2} \rightarrow 4$. Thus we write $\lim _{x \rightarrow 2}(x-1)^{2}=4$. 4 is said to be the limit of the expression $(x-1)^{2}$ as $x$ approaches 3.

## Intuitive Limit Definition

If, as $x$ approaches $b$ from both the right and the left, $f(x)$ approaches the single real number $L$, then $L$ is called the limit of the function $f(x)$ as $x$ approaches $b$, and we write $\lim _{x \rightarrow \infty} f(x)=L$.


Ex \#2: Using the given graph, calculate each limit:

$x$
(a) $\lim _{x \rightarrow 2} f(x)$
(b) $\lim _{x \rightarrow 0} f(x)$
(c) $\lim _{x \rightarrow-2.5} f(x)$
(d) $\lim _{x \rightarrow 1} f(x)$
(e) $f(1)$
(f) $f(-2)$
(g) $\lim _{x \rightarrow-2} f(x)$
(h) $\lim _{x \rightarrow-2} f(x)$
(i) $\lim _{x \rightarrow-2} f(x)$

Ex \#3: Using the given graph, calculate each limit:
(a) $f(-6)$
(b) $f(0)$
(c) $f(3)$
(d) $f(-4)$
(e) $f(-3)$
(f) $\lim _{x \rightarrow 1} f(x)$
(g) $\lim _{x \rightarrow-5} f(x)$
(h) $\lim _{x \rightarrow 0} f(x)$
(i) $\lim _{x \rightarrow 3} f(x)$
(i) $\lim _{x \rightarrow-4^{+}} f(x)$
(k) $\lim _{x \rightarrow-4} f(x)$
(1) $\lim _{x \rightarrow-4} f(x)$
(m) $\lim _{x \rightarrow 2^{+}} f(x)$
(i) $\lim _{x \rightarrow 2^{-}} f(x)$
(o) $\lim _{x \rightarrow 2} f(x)$
(p) $\lim _{x \rightarrow-2^{+}} f(x)$
(a) $\lim _{x \rightarrow-2^{-}} f(x)$
(r) $\lim _{x \rightarrow-2} f(x)$
(s) $\lim _{x \rightarrow \infty} f(x)$
(t) $\lim _{x \rightarrow-\infty} f(x)$


NOTE: If a limit goes to either $\pm \infty$, the BEST answer (and the one I expect) will be to first show that it goes to either $\pm \infty$ and THEN conclude that the limit DNE for that reason. IF you just say $\pm \infty$ or just say DNE you will not get full points. If the limit DNE because the limits on either side of an asymptote change between $\pm \infty$, for full marks you need to show that the limit from the left does not equal the limit from the right and then conclude the limit DNE.

Ex \#4: Without using a graph, use your scientific calculator to fill in the table. Use the table to predict $\lim _{x \rightarrow 2} \frac{x^{3}-8}{x-2}$
https://www.desmos.com/calculator/y671llbewf

| $\mathbf{x}$ | $\frac{x^{3}-8}{x-2}$ |
| :---: | :---: |
| 1.9 |  |
| 1.999 |  |
| 1.9999 |  |
| 2 |  |
| 2.0001 |  |
| 2.001 |  |
| 2.01 |  |

## Ex \#5:

Below is a table of values of a rational function. Use the table to find the limits that follow.

| $x$ | -1000 | -1.001 | -1 | -0.999 | 0 | 1.999 | 2 | 2.001 | 1000 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $F(x)$ | 1.002 | 2001 | Undefined | -1999 | -1 | 0.333 | Undefined | 0.334 | 0.998 |

a) $\lim _{x \rightarrow-\infty} f(x)$
b) $\lim _{x \rightarrow-1^{-}} f(x)$
c) $\lim _{x \rightarrow-1^{+}} f(x)$
d) $\lim _{x \rightarrow 2} f(x)$
e) $\lim _{x \rightarrow \infty} f(x)$
f) $\lim _{x \rightarrow-1} f(x)$

Ex \#6: Graph the following piecewise function on your graphing calculator and use the graph to find the limits below. If you forget how to do this, see page 15 of your chapter 1 notes!
$f(x)=\left\{\begin{array}{cc}x+8 & x<-1 \\ x^{2} & -1 \leq x \leq 2 \\ 1-x & x>2\end{array}\right.$
a) $\lim _{x \rightarrow-1} f(x)$
b) $\lim _{x \rightarrow 0} f(x)$
c) $\lim _{x \rightarrow \infty} f(x)$
d) Could you use algebra to find the answers to $a$ and $b$ ?

## DEFINITION OF A LIMIT

- If, as $x$ approaches $b$ from both the right and the left, $f(x)$ approaches the single real number $L$, then $L$ is called the limit of the function $\mathrm{f}(\mathrm{x})$ as x approaches b and we write
$\lim _{x \rightarrow b} f(x)=L$
- In order for $\lim _{x \rightarrow b} f(x)$ to exist, the limit as you approach b from either side must be the same number. That is that $\lim _{x \rightarrow b^{+}} f(x)=\lim _{x \rightarrow b-} f(x)=L$ where $L$ is a real number
- If $\lim _{x \rightarrow b^{+}} f(x) \neq \lim _{x \rightarrow b-} f(x)$ then we say that "THE LIMIT DOES NOT EXIST" or DNE
- If the limit of the graph at a value b seems to approach infinity, we can say that $\lim _{x \rightarrow b} f(x)=\infty$ even though $\infty$ itself is not defined as a real number. Technically, the correct answer to write when a limit is $\infty$ is "the limit does not exist". However, since writing that the limit is $\infty$ is sometimes used as it describes more clearly the manner in which the limit does not exist.


### 2.1 Day 2 Assignment: DUO TANG Ch 2 P3-4 \# 1, 2, 8-1 3 \& Textbook P66 \#35, 43-50, 57, 58, 60

### 2.1 Day 3 - Solving Limits Using Algebra

VIDEO LINKS:
a) https://bit.ly/2NOrlhv
b) https://goo.gl/MSqw2e
c) https://goo.gl/e3nNb3
d) https://goo.gl/vwy4AF

- When we don't have the graph of the function that we are finding the limit of, we need to use algebraic techniques in order to find the limit
- Today you will learn 5 different techniques. Sometimes only one of the five methods will work and sometimes more than one will work (in that case you want to work to try and find the most efficient method).


## METHOD 1: SUBSITUTION

- This method involves directly substituting the value that the variable is approaching into the expression
- This method should always be the first thing you try (but you can't use it if the value that the variable is approaching is itself a non-permissible value of the expression)

Ex \#1: Find the following limits: $\qquad$ https://www.desmos.com/calculator/6lzuwibgm3
a) $\lim _{x \rightarrow 3}\left(x^{2}-5 x+4\right)$
b) $\lim _{x \rightarrow-3} \frac{x+9}{x-3}$
c) $\lim _{x \rightarrow 9} \frac{\log _{3} x}{\sin \left(\frac{\pi x}{18}\right)}$
d) $\lim _{\theta \rightarrow \frac{2 \pi}{3}} \frac{\cos \theta}{\theta}$

## METHOD 2: FACTOR AND REDUCE

- If you have a rational function, you may be able to factor the numerator and denominator and reduce the function by cancelling. At that point you may be able to use the first method of SUBSTITUTION to find the limit.

Ex \#2: Find the following limits:
a) $\lim _{x \rightarrow 5} \frac{x^{2}-6 x+5}{x-5}$
b) $\lim _{x \rightarrow 1} \frac{10 x^{2}-10 x}{x^{3}-1}$

## METHOD 3: SIMPLIFYING

- If direct substitution leaves you with a zero in the denominator and you can't factor, apply your knowledge of adding/subtracting/multiplying/dividing fractions until you simplify the expression into one that substitution will work to find the limit

Ex \#3: Find the following limits:
a) $\lim _{x \rightarrow 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$
b) $\lim _{h \rightarrow 0} \frac{(-2+h)^{3}-2(-2+h)+4}{h}$

## METHOD 4: RATIONALIZING

- If your function has radicals in either its numerator or its denominator, rationalize to remove the radical by multiplying the numerator and denominator by the CONJUGATE

Ex \#4: Find the following limits:
a) $\lim _{r \rightarrow 6} \frac{\sqrt{3+r}-3}{r-6}$
b) $\lim _{h \rightarrow 6} \frac{6-h}{\sqrt{10-h}-\sqrt{h-2}}$

## METHOD 5: SIGN ANALYSIS

- This method will work with rational functions IF the number that the variable is approaching is ALSO THE LOCATION OF AN ASYMPTOTE of the function - ie we are approaching some number $a$ and there is a vertical asymptote at $x=a$.
- When you find $f(a)$ and get $\frac{k}{0}$ OR you end up with $\frac{k}{0}$ after factoring/canceling, then $\lim _{x \rightarrow a} f(x)$ does not exist. (This means that we wouldn't be able to use substitution in this situation because the value that the variable is approaching is also a non-permissible value and will produce a zero in the denominator )
- In order to find the limit, we need to find out how the graph is behaving on either side of the asymptote we don't actually need any specific value to find the behavior, just the sign. We perform a sign analysis of the function $f(x)$ to see if the graph is approaching $+\infty$ or $-\infty$ on either side of the asymptote
- If the sign analysis shows that the function is approaching $+\infty$ on both sides of $a$, we can say that $\lim _{x \rightarrow a} f(x)=+\infty, \therefore$ Does Not Exist . NOTE: We use this definition because it gives us a good image of how the graph looks, but technically $+\infty$ is not a defined limit because $+\infty$ is a concept, not a NUMBER (limits are defined to be a REAL NUMBER L)
- If the sign analysis shows that the function is approaching $-\infty$ on both sides of $a$, we can say that $\lim _{x \rightarrow a} f(x)=-\infty, \therefore$ Does Not Exist
- If the sign analysis shows that one side is approaching $+\infty$ and one side approaching $-\infty$, the limit doesn't exist because $\lim _{x \rightarrow a^{-}} f(x) \neq \lim _{x \rightarrow a^{+}} f(x)$

Ex \#5: Find the following limits:
a) $\lim _{x \rightarrow 2} \frac{x}{x^{2}-4}$
https://www.desmos.com/calculator/r8jd3qws4q
b) $\lim _{x \rightarrow 0} \frac{x+3}{x^{4}-4 x^{3}-21 x^{2}}$
and

$$
\lim _{x \rightarrow 7} \frac{x+3}{x^{4}-4 x^{3}-21 x^{2}}
$$

## LIMIT PROPERTIES

If $c$ and $k$ are real numbers, $n$ is an integer, $\lim _{x \rightarrow c} f(x)$ is a real number, and $\lim _{x \rightarrow c \in c} g(x)$ is a real number, then:

1. If $f(x)$ is the constant function $f(x)=k$, then $\lim _{x \rightarrow c} k=k$.
2. If $f(x)$ is the identity function $f(x)=x$, then $\lim _{x \rightarrow c} x=c$.
3. Sum: $\lim _{x \rightarrow 0}[f(x)+g(x)]=\lim _{x \rightarrow \infty} f(x)+\lim _{x \rightarrow c} g(x)$.
4. Difference: $\lim _{x \rightarrow 0}[f(x)-g(x)]=\lim _{x \rightarrow \infty} f(x)-\lim _{x \rightarrow 0} g(x)$.
5. Product: $\lim _{x \rightarrow c}[f(x) \cdot g(x)]=\lim _{x \rightarrow c} f(x) \cdot \lim _{x \rightarrow e} g(x)$.
6. Quotient: $\lim _{\delta \rightarrow c}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim _{x \rightarrow e} f(x)}{\lim _{x \rightarrow c} g(x)}$, provided $\lim _{x \rightarrow \pi} g(x) \neq 0$.
7. Constant Multiple: $\lim _{x \rightarrow 0}[k \cdot f(x)]=k \cdot \lim _{x \rightarrow 0} f(x)$ (kis a real number).
8. Power: $\lim _{x \rightarrow \infty}[f(x)]^{n}=\left[\lim _{x \rightarrow e} f(x)\right]^{n}$, $(n$ is a positive integer).
9. Root: $\lim _{x \rightarrow \infty} \sqrt[n]{f(x)}=\sqrt[n]{\lim _{x \rightarrow C} f(x)}$, provided the root exists.

Put into words, 3 to 9 say:
3. The limit of a sum is the sum of the limits.
4. The limit of a difference is the difference of the limits.
5. The limit of a product is the product of the limits.
6. The limit of a quotient is the quotient of the limits, provided the denominator is not 0 .
7. The limit of a constant multiple of a function is the constant fimes the limit of the function.
8. The limit of a positive integral power of a function is the power of the limit of the function.
9. The limit of a root of a function is the root of the limit, provided the root exists.

Ex \#6: Graph of $f(x)$


Graph of $g(x)$


Find each of the following limits applying the properties of limits. If a limit does not exist, state why.

| $\lim _{x \rightarrow 2^{-}}[f(x)+g(x)]$ | $\lim _{x \rightarrow-1}[2 f(x)-3 g(x)]$ | $\lim _{x \rightarrow-3}[f(x)-g(x)]$ |
| :---: | :---: | :---: |
| $\lim _{x \rightarrow 6} \frac{-2 f(x)}{g(x)}$ | $\lim _{x \rightarrow 4} 2[f(x) g(x)]$ | $\lim _{x \rightarrow-2}[f(x)]^{2}$ |
| $\lim _{x \rightarrow 2^{+}} \sqrt{2 g(x)}$ |  |  |


| If $g(x)=3 x+2$, find $\lim _{x \rightarrow 0} \frac{2 g(x)+g(-2)}{x}$. | If $f(x)=2 x^{2}-3 x+4$, find $\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$. |
| :--- | :--- |

Ex \#7: Write the equation of the piece-wise defined function pictured to the right.



Use the equation that you just wrote to find each of the following limits. Confirm
your results based on the graph. If a limit does not exist, state why.
a) $\lim _{x \rightarrow 2^{+}} f(x)$
b) $\lim _{x \rightarrow 2} f(x)$
c) $\lim _{x \rightarrow-3^{-}} f(x)$
d) $\lim _{x \rightarrow-3^{+}} f(x)$
e) $\lim _{x \rightarrow-7} f(x)$
f) $\lim _{x \rightarrow-1} f(x)$

### 2.1 Day 4 - Limits Involving Trig Functions

Using what is called THE SQUEEZE THEOREM (also called the Sandwich Theorem or Pinching Theorem) we can develop some rules that allow us to take the limit of particular trig functions where substitution gives us indeterminate values, and where algebraic techniques don't apply

If $h(x) \leq f(x) \leq g(x)$ for all $x$ in an open interval containing $c$, except possibly at $c$ itself, and if

$$
\lim _{x \rightarrow c} h(x)=L=\lim _{x \rightarrow c} g(x)
$$

then $\lim _{x \rightarrow c} f(x)$ exists and is equal to $L$.


## One example of applying the squeeze theorem to Calculus:

Given the following unit circle where $\theta$ is measured in radians

- Find the area of triangle POA ( $\triangle P O A$ )
- The coordinates of point P are $(\cos \theta, \sin \theta) \therefore$ the Area of $\triangle P O A=\frac{1}{2}(O A)(A P)=\frac{1}{2} \cos \theta \sin \theta$
- Find the area of SECTOR POB
- The area of a sector with radius $r$ and central angle $\theta$ in radians is $A=\frac{1}{2} r^{2} \theta$
- Since this is a unit circle, $\mathrm{r}=1$ and the area will be $A=\frac{1}{2}(1)^{2} \theta=\frac{1}{2} \theta$
- Find the area of $\triangle Q O B$
- The area of $\triangle Q O B$ is $A=\frac{1}{2}(O B)(Q B)$ where $O B=1, \therefore A=\frac{1}{2} Q B$
- $\triangle Q O B$ is a right triangle and therefore $\tan \theta=\frac{Q B}{O B}$ where $\mathrm{OB}=1, \therefore \tan \theta=Q B$

- Finally, the area of $\triangle Q O B$ will be $A=\frac{1}{2}(O B)(Q B)=\frac{1}{2}(1) \tan \theta=\frac{1}{2} \tan \theta$
- Now Putting it all together
$>$ Visually we can see that the following is true: Area of $\triangle \mathrm{POA}<$ Area of Sector $\mathrm{POB}<$ Area of $\triangle Q O B$
> Writing this mathematically we have:

$$
\begin{aligned}
& \frac{1}{2} \cos \theta \sin \theta<\frac{1}{2} \theta<\frac{1}{2} \tan \theta \\
& \cos \theta \sin \theta<\theta<\tan \theta \\
& \cos \theta<\frac{\theta}{\sin \theta}<\frac{1}{\cos \theta} \\
& \frac{1}{\cos \theta}>\frac{\sin \theta}{\theta}>\cos \theta
\end{aligned}
$$

> Multiplying each inequality by two we will have:
> Dividing each inequality by $\sin \theta$ we will have
$>\frac{\sin \theta}{\theta}$ is squeezed in between $\cos \theta$ and $\frac{1}{\cos \theta}$, both of which approach 1 as $\theta \rightarrow 0$. Thus by the squeeze theorem we can say that $1>\frac{\sin \theta}{\theta}>1$, or that $\frac{\sin \theta}{\theta}$ must be between one and one, meaning that $\lim _{\theta \rightarrow 0} \frac{\sin \theta}{\theta}=1$

Ex \#1: Apply the squeeze theorem to evaluate $\lim _{x \rightarrow 0} x \cos x$. https://www.desmos.com/calculator/4h9nnliplt


## IMPORTANT LIMITS OF TRIGONOMETRIC FUNCTIONS:

a) $\lim _{\theta \rightarrow 0} \frac{\sin \theta}{\theta}=1$
b) $\lim _{\theta \rightarrow 0} \frac{\theta}{\sin \theta}=1$
c) $\lim _{x \rightarrow 0} \frac{\sin k x}{k x}=1$
d) $\lim _{x \rightarrow 0} \frac{k x}{\sin k x}=1$
e) $\lim _{\theta \rightarrow 0} \frac{\cos \theta-1}{\theta}=0$
f) $\lim _{\theta \rightarrow 0} \frac{1-\cos \theta}{\theta}=0$

Ex \#2: Use the above rules to assist in finding the following limits. Note: At this point I need you to show me the algebraic manipulations used to write the finalized equivalent limit equations.
a) $\lim _{x \rightarrow 0} \frac{\sin 3 x}{x}$
b) $\lim _{x \rightarrow 0} \frac{\sin 5 x}{\sin 7 x}$
c) $\lim _{x \rightarrow 0} \frac{\tan x}{2 x}$
d) $\lim _{x \rightarrow 0} \frac{x^{2}}{\tan x}$
e) $\lim _{x \rightarrow 0} \frac{1-\cos x}{\tan x}$
f) $\lim _{x \rightarrow 0} \frac{1-\cos ^{2} x}{x}$
g) $\lim _{x \rightarrow 0} \frac{2 \sin x-\sin 2 x}{x \cos x}$

## LIMITS OF COMPOSITE FUNCTIONS

The commonly memorized limit of composition rule generally is presented as something like this:

$$
\lim _{x \rightarrow a} f(g(x))=f\left(\lim _{x \rightarrow a} g(x)\right) \text { provided that } \lim _{x \rightarrow a} g(x)=L \text { and } f(x) \text { is continuous at } x=L .
$$

- This is true (and easy to use if the last two statements are true
- If there are issues with the last two statements we can't apply the rule as stated but we don't know anything conclusive about $\lim _{x \rightarrow a} f(g(x))$

Ex \#3: Functions $g$ and $h$ are graphed.
Find $\lim _{x \rightarrow-3} g(h(x))$.



## Ex \#4:

$$
\text { Find } \lim _{x \rightarrow 0} f(h(x))
$$




## Ex \#5:




1. $\quad$ Find $f(g(4))$.
2. Find $f(g(0))$.
3. Find $\lim _{x \rightarrow 4} f(g(x))$.
4. Find $\lim _{x \rightarrow 0} f(g(x))$.
5. Find $\lim _{x \rightarrow 6^{-}} g(1-f(x))$.
6. Find $\lim _{x \rightarrow 3^{+}} f\left(15-x^{2}\right)$.
7. Find $\lim _{x \rightarrow 2^{-}} g\left(2-x^{2}\right)$.
8. Find $\lim _{x \rightarrow-2^{-}} f(f(x))$
A) https://bit.ly/2LhsqBU
B) https://bit.ly/2KOR6Bs
C) $\mathrm{https}: / / \mathrm{bit} . \mathrm{ly} / 1 \mathrm{t} 8 \mathrm{dbj} 9$

D )http://bit.ly/20TPbIP
Khan Academy Links:
https://www.youtube.com/watch?v=mZiPdyHyUvE\&feature=share
https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-new/ab-1-8/v/sinx-over-x-as-x-approaches0?fbclid=IwAR1LnFWSet9rUWBHTdbsItXIX7MV8sSyu3IRtCfnZ_y1IkIENsys94zwAgk
2.1 Day 4 Assignment: Textbook P66 \#17, 18, 20, 31-33, 57, 59, 65-68 (Show all steps!) Duo Tang Ch 2 P7 Part A: \#1, 2 \& 3
Leave until Chapter 9:Duo Tang Ch 2 Part B: any of the additional questions as needed (Show all steps!)


### 2.2 Day 1 - Limits at Infinity

VIDEO LINKS:
A) https://bit.ly/2NNaaCa
B) https://goo.gl/kY6D4t

## C) https://goo.gl/5RLxzV

The Symbol for infinity $\infty$ does not represent a real number. We use $\infty$ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds.

- For example, when we say "the limit of $f$ as $\mathbf{x}$ approaches infinity" we mean the limit of $f$ (or the height of the $y$ value) as $x$ moves increasingly far to the right on the number line.
- When we say "the limit of $f$ as $x$ approaches negative infinity $(-\infty)$ " we mean the limit of $f$ (or the height of the $y$ value) as $x$ moves increasingly far to the left on the number line.

Ex \#1: Intuitively find the following limits. Imagine placing numbers close to negative or positive infinity into $x$ and evaluating. What limits do you get?
(a) $\lim _{x \rightarrow \infty} x$
(b) $\lim _{x \rightarrow \infty} x^{2}$
(c) $\lim _{x \rightarrow-\infty} x$
(d) $\lim _{x \rightarrow-\infty} x^{2}$

- Now picture the graph of each function - what does each limit represent? https://www.desmos.com/calculator/tfg2xntbef

$$
\begin{aligned}
& \text { PATTERNS: } \\
& \begin{aligned}
\lim _{x \rightarrow-\infty} x^{n} & : \text { if } n \text { is even } \rightarrow \lim _{x \rightarrow-\infty} x^{n}=\infty \\
& : \text { if } n \text { is odd } \rightarrow \lim _{x \rightarrow-\infty} x^{n}-\infty \\
\lim _{x \rightarrow-\infty}- & x^{n}: \text { if } n \text { is even } \rightarrow \lim _{x \rightarrow-\infty} x^{n=}-\infty \\
& : \text { if } n \text { is odd } \rightarrow \lim _{x \rightarrow-\infty} x^{n}=\infty
\end{aligned}
\end{aligned}
$$

Ex \#2: Given that $f(x)=\frac{x+1}{x}$, use a graph and tables to find the following:
a) $\lim _{x \rightarrow \infty} f(x)$
b) $\lim _{x \rightarrow-\infty} f(x)$
c) Identify all horizontal asymptotes



In pre-calculus, we learned three rules for determining the existence of horizontal asymptotes of rational functions. When a rational function had a horizontal asymptote, the end behavior was always such that as $x \rightarrow-\infty$ or $\infty$, then the graph of $f(x) \rightarrow$ the horizontal asymptote. We learned three rules for determining the horizontal asymptote, if one existed, for rational functions. We are about to use the idea of a limit and calculus to find out why those rules are such as they are. For each function below, divide every term in both the numerator and the denominator by the highest power of $x$ that appears in the denominator. Then, evaluate the indicated limit. Does the result of each limit make sense based on the graph that is pictured?

## Ex \#3:

a) $\lim _{x \rightarrow \infty} \frac{2 x^{2}+7 x+6}{x^{2}+5 x+6}$

b) $\lim _{x \rightarrow \infty} \frac{x^{2}+3 x+2}{x-1}$


Would your answers be different if we were asked to approach negative infinity?

The line $y=b$ is a horizontal asymptote of the graph of a function $y=f(x)$ if either

$$
\lim _{x \rightarrow \infty} f(x)=b \quad \text { or } \quad \lim _{x \rightarrow-\infty} f(x)=b
$$

The sandwich theorem also holds for limits as $x \rightarrow \pm \infty$.

Ex \#4: Find $\lim _{x \rightarrow \infty} \frac{\cos x}{x}$ graphically and using a table of values. https://www.desmos.com/calculator/gzahskg7yg

- Both the graph and table suggest that the function oscillates about the
$\qquad$ _.
- Therefore the horizontal asymptote is at


$\qquad$
- This means that $\lim _{x \rightarrow \infty} \frac{\cos x}{x}=$ $\qquad$

Ex \#5: Using example 2 from above, what is the equation of the vertical asymptote?

What limit equation would represent this asymptote? Why?

## DEFINITION Vertical Asymptote

The line $x=a$ is a vertical asymptote of the graph of a function $y=f(x)$ if either

$$
\lim _{x \rightarrow a^{+}} f(x)= \pm \infty \quad \text { or } \quad \lim _{x \rightarrow a^{-}} f(x)= \pm \infty
$$

## QUESTION:

- What happens if I take a number such as 6 , and continually try dividing it by larger and larger numbers?

$$
\text { What will the answer eventually approach? } \frac{6}{2}, \frac{6}{4}, \frac{6}{6}, \frac{6}{8}, \ldots \ldots \frac{6}{1000}, \frac{6}{100000000}, \ldots, \frac{6}{\infty}
$$

- What happens if I take the same question but square all of the numbers on the bottom?

$$
\frac{6}{2^{2}}, \frac{6}{4^{2}}, \frac{6}{6^{2}}, \frac{6}{8^{2}}, \ldots \ldots \frac{6}{1000^{2}}, \frac{6}{100000000^{2}}, \ldots, \frac{6}{\infty^{2}}
$$

## DIVIDING BY INFINITY

- Any number that is divided by a very large number (like $\infty$ ) will get so close to zero that we may as well say it is equal to zero
- $\lim _{x \rightarrow \infty} \frac{1}{x}=0 \quad$ (additionally this is true for higher powers of $\mathrm{x}: \lim _{x \rightarrow \infty} \frac{1}{x^{2}}=0, \lim _{x \rightarrow \infty} \frac{1}{x^{3}}=0, \ldots$ )

Ex \#6: Find the following limits. Explain what the result of the limit means about the graph of each rational function.
(a) $\lim _{x \rightarrow \infty} \frac{2 x^{2}+5 x+2}{x^{2}-9}$
https://www.desmos.com/calculator/7vwykqg69u
(b) $\lim _{x \rightarrow \infty} \frac{4 x^{3}-2 x^{2}}{6-5 x}$
(c) $\lim _{x \rightarrow-\infty} \frac{x-3}{x^{2}+5 x-12}$

Ex \#7: The function $f$ is shown below. Determine the equations of all vertical and horizontal asymptotes of $f$ and the equivalent limit equations.


FINDING THE LIMIT OF A RATIONAL EXPRESSION AS $x \rightarrow \infty$ or $x \rightarrow-\infty$

- Multiply the numerator and denominator by the reciprocal variable with the highest degree in the denominator, simplify and apply the limit.
- If the degree of the numerator is the same as the degree of the denominator, the answer to the limit as $x \rightarrow \pm \infty$ will be equal to $\qquad$
- If the degree of the numerator is less than the degree of the denominator, the answer to the limit as $x \rightarrow \pm \infty$ will be equal to
- If the degree of the numerator is greater than the degree of the denominator, the answer to the limit as $x \rightarrow \pm \infty$ will be equal to $\qquad$


### 2.2 Day 2 - Limits at Infinity Involving Radicals

## VIDEO LINKS: A) https://bit.ly/2KVQYo2 <br> B) https://goo.gl/kY6D4t <br> C) https://goo.gl/5RLxzV

FINDING THE LIMIT OF A RADICAL EXPRESSION AS $x \rightarrow-\infty$ or $x \rightarrow-\infty$

- NOTE: Because $\sqrt{x^{2}}= \pm x$, we can really say that $\sqrt{x^{2}}=|x|$
- With a radical expression we need to factor out a GCF from under the root sign that you will be able to take the exact root of. If there is a non radical numerator or denominator, you need to take out a GCF of the highest power of its variable
- You need to consider whether the question is asking for $x \rightarrow \infty$ or $x \rightarrow-\infty$ as this will direct you as to whether you are using $\pm x$ when the question has a $\sqrt{x^{2}}$ or a $|x|$.

$$
\text { - } \lim _{x \rightarrow \infty}|x|=+x \quad \text { and } \quad \lim _{x \rightarrow-\infty}|x|=-x
$$

Ex \#1: Find the following limits:
a) $\lim _{x \rightarrow \infty} \frac{\sqrt{x^{2}+x-2}}{x+1}$
b) $\lim _{x \rightarrow-\infty} \frac{x^{2}+2 x}{\sqrt{3 x^{2}+2}}$
c) $\lim _{x \rightarrow \infty} \frac{x^{2}+4}{\sqrt{4 x^{4}+x^{2}+1}}$

Ex \#2: Find the following limit at infinity. What do the results show about the existence of a horizontal asymptote?
Justify your reasoning.
$\lim _{x \rightarrow-\infty} \frac{3 x-2}{\sqrt{2 x^{2}+1}}$
2.2 Day 2 Assignment: Textbook P 76 \#35-38

## 2.3-Continuity

To learn what is meant by a continuous function and to learn about the four different types of discontinuities..

## VIDEO LINKS:

(NOTE: The last three video's only speak about three different types of discontinuities - it does not discuss oscillating discontinuity)
A) https://bit.ly/2KXD1pO
B) https://goo.gl/xkXnq3
C) https://goo.gl/ryNScH
D) https://goo.gl/F6x416

## CONTINUOUS FUNCTIONS: Formal Definition

- A function is considered to be continuous at a specific $x$ value of "a" if the following is true:
$\lim _{x \rightarrow a^{+}}=\lim _{x \rightarrow a^{-}}=f(a)$
The right limit at $\mathrm{a}=$ The left limit at $\mathrm{a}=$ The actual height of the function at a
- Possible Problems:






## THERE ARE FOUR DIFFERENT WAYS THAT A FUNCTION CAN BE DISCONTINUOUS:

1. INFINITE DISCONTINUITY

- This is where the graph has a vertical asymptote and where the limit of a graph approaches $\infty$ or $-\infty$
- They can be found algebraically by finding where the values of $x$ where the denominator of a rational function will be zero


## 2. REMOVABLE DISCONTINUITY

- On the graph there will be a hole
- This happens in equations where a factor in the numerator cancels with a factor in the denominator
- The ordered pair of the hole can be found algebraically by cancelling the common factor and then substituting the $x$ value of that factor into the resulting equation



Oscillating Discontinuity

## Continuity Principles

1. All constant functions are continuous.
2. The following types of functions are continuous at every point in their domain: polynomial, rational, power, root, trigonometric, exponential, and logarithmic,
3. If $f(x)$ and $g(x)$ are continuous functions, so are
$(f+g)(x) \cdot(f-g)(x),(f g)(x)$, and $\left(\frac{f}{g}\right)(x)$ in their common domains provided, in the last case, that $g(x) \neq 0$.

## DEFINITION Continuity at a Point

Interior Point: A function $y=f(x)$ is continuous at an interior point $c$ of its domain if

$$
\lim _{x \rightarrow c} f(x)=f(c) .
$$

Endpoint: A function $y=f(x)$ is continuous at a left endpoint $a$ or is continuous at a right endpoint $b$ of its domain if

$$
\lim _{x \rightarrow a^{+}} f(x)=f(a) \quad \text { or } \quad \lim _{x \rightarrow b^{-}} f(x)=f(b), \quad \text { respectively }
$$

In addition, the composites of continuous functions are continuous. If $f(x)$ and $g(x)$ are continuous, $f(g(x))$ and $g(f(x))$ are continuous.

Ex \#1: Determine whether or not the following functions are continuous. If it is not continuous, identify where the discontinuity occurs and classify the discontinuity.
a) $f(x)=\frac{x^{2}-4}{x-2}$
b) $f(x)=\left\{\begin{array}{l}x^{2}, x \neq 2 \\ 1, x=2\end{array}\right.$
c) $f(x)=\left\{\begin{array}{l}x+1, x \leq 2 \\ \frac{x-5}{x-3}, x>2\end{array}\right.$
d) $f(x)=\frac{\left|x^{2}-3 x+2\right|}{x-2}$
e) $f(x)=\sqrt{x-25}$
f) $f(x)=\sqrt[3]{x-5}$

Ex \#3: Given the function $f(x)=\frac{x^{3}-6 x^{2}+8 x}{x^{2}-5 x+6}$
a) Where is the function discontinuous?
b) What value would you need to assign to make it continuous (piecewise), if possible?

Ex \#4: Use the three-part definition of continuity to determine if the given functions are continuous at the indicated values of $x$.
a. $f(x)=\left\{\begin{array}{cc}-2 \sqrt{x+6}, & x<-2 \\ 3 x+2, & x=-2 \\ e^{x}+\cos (\pi x), & x>-2\end{array}\right.$ at $x=-2 \quad$ b) $g(x)=\left\{\begin{array}{cc}e^{x} \cos x, & x<\pi \\ e^{x} \tan \left(\frac{3 x}{4}\right), & x \geq \pi\end{array}\right.$ at $x=\pi$
$\begin{aligned} & \text { Ex \#5: Consider the function, } f(x) \text {, to the right to answer the following questions. } \\ & \text { a. What two limits must equal in order for } f(x) \text { to be continuous at } x=-1 \text { ? }\end{aligned} f(x)=\left\{\begin{array}{cc}2, & x \leq-1 \\ m x+k, & -1<x<3 \\ -2, & x \geq 3\end{array}\right.$
b. What two limits must equal in order for $f(x)$ to be continuous at $x=3$ ?
c. Determine the values of $m$ and $k$ so that the function is continuous everywhere.

## The INTERMEDIATE VALUE THEOREM

If $f(x)$ is continuous on the closed interval $[\mathrm{a}, \mathrm{b}]$ and $f(a) \leq y_{0} \leq f(b)$, then there is at least one number $\mathrm{c}, a<c<b$ such that $f(c)=y_{0}$.

A function $y=f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if $y_{0}$ is between $f(a)$ and $f(b)$, then $y_{0}=$ $f(c)$ for some $c$ in $[a, b]$.


Ex \#6: Use the Intermediate Value Theorem to show that $x^{3}-2 x-2=0$ has a solution in the interval [1,3]

Ex \#7: Investigate the following graphs to determine if the IVT is applicable for these functions on the specified intervals for the values given.
a) $f(x)=\left\{\begin{array}{cc}-(x+3)^{2}+4, & x<-2 \\ -\frac{1}{2} x-1, & x>-2\end{array}\right.$

- Is there a value of $c$ on $[-5,2]$ such that $f(c)=2$ ?

- Does the I.V.T. guarantee a value of $c$ such that $f(c)=2$ on the interval [-5, 2]? Why or why not
b) $f(x)=\left\{\begin{array}{cc}-(x+3)^{2}+4, & x<-2 \\ -\frac{1}{2} x-1, & x>-2\end{array}\right.$
- Is there a value of $c[-1,5]$ such that $f(c)=2$ ?

- Does the I.V.T. guarantee a value of $c$ such that $f(c)=2$ on the interval $[-1,5]$ ? Why or why not?


## 2.4- Rates of Change and Tangent Lines

VIDEO LINKS: a
a) https://bit.ly/2NirEFD
b) https://goo.gl/gNkGBX
c) https://goo.gl/wCNKSW
d) https://goo.gl/NSrJFS
e) https://goo.gl/KpZWGz

TANGENT LINE - A line that touches a curve in one place

SECANT LINE - A line touches a curve in two or more places


## USING THE SLOPE OF A SERIES OF SECANT LINES TO FIND THE SLOPE OF THE TANGENT LINE: Animation Link:

https://www.google.com/search?q=secant+line+approaching+a+tangent+line\&rlz=1C1GCEB enCA811CA811\&source=|nms\&tbm=isc h\&sa=X\&ved=OahUKEwjQi4DG8NrjAhUXZcOKHYPICaUQ AUIESgB\&biw=1366\&bih=608\&safe=active\&ssui=on\#imgdii=CwHaxrlkmhbz tM:\&imgrc=K7TO04Ov WGaqM:

$\Delta x$

$\Delta x$


Secant Lines that are approaching/becoming a Tangent Line as $Q$ moves closer to $P$ and $\Delta x \rightarrow 0$
$\longleftarrow$ THESE ARE ALL INDIVIDUAL AVERAGE RATES OF CHANGE $\longrightarrow$


This is the Tangent Line where $P=Q$ and $\Delta x=0$
the tangent is the INSTANTANEOUS RATE OF CHANGE

## INTUITIVE DEFINITION OF A TANGENT LINE:

If, as we change the sequence of points $\mathrm{Q}_{n}$, the secant lines $\overleftrightarrow{P Q_{n}}$ all approach the same unique line regardless as to whether the points $Q_{n}$ are on the left or the right side of $P$, the unique line that is formed is called the TANGENT line to the curve at point $P$, and we say that the tangent line exists.

- In our above example, we only approached from the right. In the following diagram of the tangent, please draw three secants from the right in one colour and three secants from the left in another colour. Do your two colours merge at point P?


Ex \#1: Explain where the following graphs do not have tangent lines and why.


## PLACES WHERE TANGENT LINES DO NOT EXIST:

Tangent lines do not exist as places where the slope approached from the left does not equal the slope approached from the right. This occurs in places where the graph has:

A $\qquad$ or a $\qquad$

Ex \#2: The following diagram is a graph of $f(x)=x^{2}-4 x$. Sketch a tangent line at $x=3$ and $x=0$ then estimate the slope of each.
b) What is the slope when $x=2$ ?


## Ex \#3:

a) Find the slope of the secant line $P Q$.

- Estimate the slope of the tangent line drawn to the function $f(x)=x^{2}$ at the point $P(1, f(1))$ by filling in the following table..

|  | POINT | x | $f(x)$ | Slope of $\overleftrightarrow{P Q_{n}}$ |
| :---: | :---: | :---: | :---: | :---: |
|  | $\mathrm{Q}_{1}$ | 0 |  |  |
|  | $\mathrm{Q}_{2}$ | 0.5 |  |  |
|  | $\mathrm{Q}_{3}$ | 0.6 |  |  |
|  | $\mathrm{Q}_{4}$ | 0.9 |  |  |
|  | P | 1 | 1 |  |
|  | $\mathrm{Q}_{5}$ | 1.01 |  |  |
|  | Q | 1.1 |  |  |
|  | $\mathrm{Q}_{7}$ | 1.5 |  |  |
|  | Q 8 | 2 |  |  |


b) Find the slope of the tangent line by taking the limit of the slope of the secant line. This can be expressed as:

$$
\lim _{Q \rightarrow P} m_{P Q}=m \quad \text { and } \quad \lim _{x \rightarrow p} \frac{f(x)-f(1)}{x-p}=\lim _{x \rightarrow 1} \frac{x^{2}-1}{x-1}=
$$

c) Now that we know that the tangent line passes through $\mathrm{P}(2,4)$ and has slope 2 , what is the equation of the tangent line?

SLOPE OF A TANGENT LINE AT A SPECIFIC POINT (FORMULA 1) (This formula will be used more in Ch 3):

- The slope of a line tangent to a curve at a point $(\mathrm{a}, \mathrm{f}(\mathrm{a}))$ is defined as follows:
$m=\lim _{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} \quad$ (Note that this is just the slope between two points named (a, $\left.\mathrm{f}(\mathrm{a})\right) \&(\mathrm{x}, \mathrm{f}(\mathrm{x}))$

Ex \#4: Find the slope and the equation of the tangent line to the curve $y=4 x^{2}-3 x-1$ at the point $(2,9)$ by using the above slope formula.

## SLOPE OF A TANGENT LINE AT A SPECIFIC POINT (Formula 2): this is the main formula

 TO USE DURING THIS CHAPTER- The slope of a line tangent to a curve at a point $(a, f(a+h))$ is defined as follows:
- 

$m=\lim _{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}=\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{(a+h)-a}=\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}$
NOTE: This is also the slope between two points ( $\mathrm{a}, \mathrm{f}(\mathrm{a})$ ) and the $2^{\text {nd }}$ point is a distance " h " away and its ordered pair is ( $(a+h), f(a+h))$. We are pushing the " $h$ " to it's smallest value of " 0 ".



Ex \#5: a) Find the slope of the tangent line drawn to the function $f(x)=3 x-x^{2}-1$ at the point $P(2, f(2))$.

a) Find the equation of the tangent line

NORMAL LINE TO A CURVE
The NORMAL LINE to a curve at a point is the line that is perpendicular to the tangent line at that point.

## Ex \#6:

a) Find the equation to the tangent line drawn to the function $g(x)=x^{3}$ at the point $(1, g(1))$
b) Find the equation to the normal line drawn to the function $g(x)=x^{3}$ at the point (1, $\left.g(1)\right)$

