### 5.1 Extreme Values of Functions

IGAN LOGATE THE EXTREME VALUES OF A FUNGTION AND UNDERSTAND THE EXTREME VALUE THEOREM VIDEO LINKS:
a) https://bit.ly/2JYZH58
b) https://bit.ly/2FjAOOB along with
https://bit.ly/2DAoZ9q

## Definition of Extrema

Let $f$ be defined on an interval $I$ containing $c$.

1. $\quad f(c)$ is the minimum of $f$ on $I$ if $f(c) \leq f(x)$ for all $x$ in $I$.
2. $f(c)$ is the maximum of $f$ on $I$ if $f(c) \geq f(x)$ for all $x$ in $I$.

The minimum and maximum of a function on an interval are the extreme values, or extrema, of the function on the interval. The minimum and maximum of the function on an interval are also called the absolute minimum and absolute maximum on the interval. "Global" is sometimes used for "absolute".

## The Extreme Value Theorem

If $f$ is continuous on a closed interval $[\mathbf{a}, \mathbf{b}]$ then $f$ has both a maximum value and a minimum value on the interval.

## Definition of Relative (Local) Extrema

1. If there is an open interval on $f(x)$ containing $c$ on which $f(c)$ is a maximum, then $f(c)$ is called a RELATIVE MAXIMUM of $\mathrm{f}(\mathrm{x})$. This is true if and only if $f(x) \leq f(c)$ for all $x$ in some open interval containing $c$
2. If there is an open interval on $f(x)$ containing $c$ on which $f(c)$ is a minimum, then $f(c)$ is called a RELATIVE MINIMUM of $\mathrm{f}(\mathrm{x})$. This is true if and only if $f(x) \leq f(c)$ for all $x$ in some open interval containing $c$

## Note:

- To be a relative maximum, the $y$ value must be greater than or equal to "nearby" $y$ values. To be a local minimum, the $y$ value must be less than or equal to "nearby" $y$ values.
- Some textbooks will allow the endpoints in a closed interval to be considered relative max or min points and some won't.


Example \#1: Discuss how the extreme value theorem applies to each of the following diagrams: (Note: in our notes and assignments we MAY use the acronym EVT for Extreme Value Theorem but DO NOT do this on the AP Exam!!! You must write out the full words!!!
a)

b)

c)

d)


## Definition of a Critical Number

Let $f(x)$ be a function that is defined at a value of $x=c$. If $f^{\prime}(c)=0$ or $f^{\prime}(c)$ is undefined/not differentiable at $c$, then $c$ is called a CRITICAL NUMBER of $f(x)$. (It is VERY important that you note that $f(x)$ must actually be defined at $x=c$.

- This means at places where the original graph has an asymptote or a point of discontinuity, there will be no critical value).
- Critical points are always an INTERIOR point of the domain and will never be an endpoint.




## Steps to Finding CRITICAL NUMBERS/VALUES

1. State the domain of $f(x)$. Numbers that fall outside of this domain are ineligible to be critical values
2. Find the derivative of $f(x)$ in factored form - it usually also works best to move negative exponents to the denominator. Be careful - sometimes the question is actually given $f^{\prime}(x)$ instead of $f(x)$ !
3. Find the values where $f^{\prime}(x)=0$ or where $f^{\prime}(x)$ is undefined. These will occur at values that are WITHIN the domain of $f(x)$ but cause $f^{\prime}(x)$ to have a zero denominator, an imaginary root or a cusp or corner
4. State the appropriate values from step 3 and call them critical values.

## Relative Extrema Occur Only at Critical Numbers

If $f(x)$ has a relative minimum or relative maximum at $x=c$, then $c$ is a critical number of $f(x)$




EX \#1: The domain of $\mathrm{f}(\mathrm{x})$ is all real numbers. If $f^{\prime}(x)=\frac{x^{2}-4 x-5}{x^{\frac{1}{3}}}$ what are the critical numbers for $\mathrm{f}(\mathrm{x})$ ?

EX \#2: Find the critical numbers of the function $f(x)=4 x-6 x^{\frac{2}{3}}$

EX \#3: Find the critical numbers for the function $f(x)=\frac{2 x}{x-3}$

EX \#4: Find any critical numbers of the function $f(x)=3 \sin (2 x)$ for $0 \leq x \leq 2 \pi$.

## STEPS TO FINDING ABSOLUTE EXTREMA OF A CONTINUOUS FUNCTION f(x) ON A CLOSED INTERVAL [a, b].

1. Identify the domain of $f(x)$ - remember that numbers that are NOT in the domain can NOT be critical values. If such a solution appears to be a critical number, you must deem it inadmissible.
2. Find the critical numbers of the function $f(x)$ on the interval $(a, b)$.
3. Evaluate the function at each critical number in (a, b).
4. Evaluate the function at each endpoint of the interval [a, b]
5. The greatest of the values in steps 3 and 4 is the absolute maximum. The least of these values is the absolute minimum
6. Sentence looks like:

On state closed interval here , $f(x)$ has an absolute maximum of maximum height from step 5 here when of $\quad x$ value for maximum height from step 5 here. $f(x)$ has an absolute minimum of minimum height from step 5 here when of x value for minimum height from step 5 here.

EX \#5: Find the absolute minimum and maximum values of $f(x)=3 x^{\frac{2}{3}}$ on the interval $[-1,2]$.

EX \#6: Without drawing a graph, find the absolute extrema of the function $f(x)=5 x-20$ on the interval $[-6,10]$

EX \#7: Without drawing a graph, find the absolute extrema of the function $f(x)=x^{3}-12 x$ on the interval [-5, 3].

EX \#8: Sketch the graph of a function that is continuous on the interval $[-2,3]$ and has:
(a) a global maximum of 3 , a global minimum of 1 , and no relative extrema.
(b) a relative maximum value of 2 at $x=1$, a relative minimum value of 1 at $x=0$ and no other relative extrema; a global minimum of 0 at the right endpoint of the interval and a global maximum of 3 at the left endpoint of the interval.
(c) a relative and global maximum value of 3 at $x=1$ and a relative and global minimum value of 1 at $x=-1$.
(d) a critical number at $x=0$ but no relative maximum or minimum value.





EX \#9: Sketch the graph of a function on the interval $[-2,3]$ that has:
(a) a global maximum but no relative maximum.
(b) no global maximum and no global minimum.
(c) a relative maximum and a relative minimum but no global maximum and no global minimum.




EX \#10:The derivative of a function $f(x)$ is $f^{\prime}(x)=(3-x)^{2}(x+5)$. At what value(s) of $x$ does the graph of $f(x)$ have a relative maximum? Justify your answer

## Calculius ${ }^{5}$ Day 1 The Mean Value Theorem and *Rolle's Theorem *(when time permits)

## I GAN USE THE MEAN VALUE THEORM FOR DERIVATIVES AND ROLLES THEOREM

VIDEO LINKS:
a) https://bit.ly/2RW8o39
b) https://bit.ly/20F3F3F

- Recall from earlier lessons what we mean by average slope:
- We often used this idea when finding average speed.
- Instantaneous speed would be the slope of a tangent line rather than a secant line
- Remember from statistics that mean means average

$$
\frac{\Delta x}{\Delta y}=\frac{f(a)-f(b)}{a-b}
$$



- Think abocut this for a second: If you are travelling from time $a$ to time $b$, and you use the secant line between those two points, you can find the average speed you are going.
- Question: Do you think there is always going to be at least one point where your instantaneous speed is equal to the average speed?


## MEAN VALUE THEOREM FOR DERIVATIVES

If $y=f(x)$ is continuous at every point of the closed interval $[\mathrm{a}, \mathrm{b}]$ and differentiable at every point of its interior $(a, b)$, then there is at least one point $c$ in $(a, b)$ at which

$$
f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}
$$

- In other words, if you have a function that meets three criteria (continuous, closed interval and differentiable on the interior of that interval), there is at LEAST one point where the instantaneous rate $=$ average rate (or the slope at the endpoints= the slope at some other point in the interval)
- This theorem guarantees the existence of a tangent line Parallel to the secant line that goes through the endpoints.

(Note that the theorem only GUARANTEES the existence of such a point - it does not FIND the point)
- This theorem implies there is a point c in the given interval where the average rate of change (slope of the secant) equals the instantaneous rate of change (slope of the tangent) f'(c)


## STEPS TO APPLYING THE MEAN VALUE THEOREM TO f(x) TO FIND THE GUARANTEED VALUE

1. Verify that $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval $(a, b)$
2. Find value of $f(a)$ and $f(b)$. Use to find the slope of the secant line from (a,f(a)) to (b,f(b))
3. Find $f^{\prime}(x)$ - do NOT factor it!
4. Solve for where the $f^{\prime}(x)$ equals the value of the slope of the secant line in step 2 . This gives you the value of $\mathrm{X}=\mathrm{c}$ that satisfies the equation $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$
5. If asked to JUSTIFY or EXPLAIN your work, the sentence you write will often look like the following (it may be different when given different initial information. On the AP Exam you may not need to write the entire sentence but all details must be included somewhere in th question. - remember to use the ACTUAL function name if it isn't $f(x)$ ):

Since $f(x)$ is continuous on write the closed interval here and differentiable on_write the open interval here the MEAN VALUE THEOREM guarantees there is a value of $x=c$ for value of a in the interval $<c<$ value of $b$ in the interval such that $f^{\prime}(c)=$ slope of the secant line from step 2 . The value of $\mathbf{x}=\mathbf{c}$ is $\mathbf{x}=$ the answer you received in step 4

## STEPS TO APPLYING THE MEAN VALUE THEOREM TO f(x) TO GUARANTEE A VALUE (NOT find it)

1. Verify that $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval $(a, b)$
2. Find value of $f(a)$ and $f(b)$. Use to find the slope of the secant line from ( $a, f(a))$ to (b,f(b))
3. You WILL need to write the following sentence (it may be different when given different initial information - remember to use the ACTUAL function name if it isn't $f(x)$ ):

Since $f(x)$ is continuous on write the closed interval here_ and differentiable on_write the open interval here the MEAN VALUE THEOREM guarantees there is a value of $x=c$ for value of a in the interval < c < value of b in the interval such that $f^{\prime}(c)=$ slope of the secant line from step 2.

Apply the Mean Value Theorem for $f(x)=x^{3}-6 x^{2}+9 x+2$,
on the interval $[0,4]$. Find all values of c in the interval such that $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$. Justify.
https://www.desmos.com/calculator/fwox86azll

EX \#2: Apply the Mean Value Theorem for $f(x)=2 \sin x+\sin (2 x)$, on the interval $[\mathrm{o}, \pi]$. Find all values of c in the interval such that $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$


EX \#3: Find the value(s) for x such that it satisfies the mean value theorem for $f(x)=x^{2} e^{x},[-5,1]$. You may use your graphing calculator. Justify.. (30L)
https://www.desmos.com/calculator/ydsr3puil1
EX \#4: Explain why the Mean Value Theorm does not apply to each of the following functions on the interval [o, 6].
a)

b)

c) $\quad f(x)=\frac{1}{x-3}$
d) $f(x)=|x-3|$

EX \#5: Consider the function $h(x)=3-\frac{5}{x}$. The graph of $h(x)$ is pictured to the right.
a) Does the M.V.T. apply on the interval $[-1,5]$ ? Explain why or why not.
b) Does the M.V.T. apply on the interval $[1,5]$ ? Why or why not?

c) Graphically, what does the M.V.T. guarantee for the function on the interval [1,5]? Draw this on the graph to the left.
d) Apply the M.V.T. to find the value(s) of $c$ guaranteed for $h(x)$ on the interval $[1,5]$

## A FEW EXTRA HINTS ABOUT THE MEAN VALUE THEOREM

- It is almost a guarantee that you will have a Mean Value Theorem Question within the Free Response part of the AP Exam, and that question will almost certainly be "disguised"
- This question may not actually ask you for a "value" answer, it may only ask whether or not a "value exists".
- The stem of this question may begin with the following:
- Is there a time. $\qquad$
- Can you guarantee that $\qquad$
- Must there be.......

NOTE: These stems may also be used for questions that deal with the OTHER existance theorems that you will learn this year.

## ROLLE'S THEOREM - This is just an extension of the Mean Value Theorem

Let $f(x)$ be a function that satisfies the following three hypotheses:
a) $f(x)$ is continuous on the closed interval [a, b]
b) $f(x)$ is differentiable on the open interval $(\mathrm{a}, \mathrm{b})$
c) $f(a)=f(b)$

Then there is at least one value of " $c$ " in $(a, b)$ such that $f^{\prime}(c)=0$

ROLLE'STHEOREM
French mathematician Michel Rolle first published the theorem that bears his name in 1691. Before this time, however, Rolle was one of the most vocal critics of calculus, stating that it gave erroneous results and was based on unsound reasoning. Later in life, Rolle came to see the usefulness of calculus.

- In other words, if the height of the graph at the beginning of the open interval equals the height at the end of the open interval, there is at least one place in between where the derivative is zero (one place that has a horizontal slope). There may be more than one such place if you have multiple max/min points or infinite places if the function itself is a horizontal line in this interval.
- This theorem gives conditions that guarantee a horizontal tangent can be found somewhere in the interval which the max or min is NOT at an endpoint

[^0]EX \#5: Given the function below, show that Rolle's Theorem applies on the interval [-4, o] and find all values of c that satisfy the theorem. $f(x)=x^{2}+4 x-5$

EX \#6: Explain why Rolle's Theorem does not apply to the following functions.
a) $f(x)=\left|\frac{1}{x}\right|$,
b) $f(x)=\cot \frac{x}{2}$,
[ $\pi, 3 \pi$ ]
$[-1,1]$
https://www.desmos.com/calculator/xtvfrpgsdv
https://www.desmos.com/calculator/cmup2fvukp
c) $f(x)=1-|x-1|$,
[0, 2 ]
https:///www.desmos.com/calculator/mjhvgeawue
d) $f(x)=\sqrt{\left(2-x^{2 / 3}\right)^{3}}$,
https:/[kwhr.desmos.com/calculator/iojsg.5u4vs

I CAN ANALYTICALLY USE THE FIRST DERIVATIVE TO DETERMINE INTERVALS WHERE A FUNCTION IS DECREASING OR INCREASING AND HAS MAXIMUM OR MINIMUM VALUES
VIDEO LINKS:
a) https://bit.ly/2T8xBbJ
b) https://bit.ly/2B18eBM
c) https://bit.ly/2PrKDmE

An intuitive explanation of increasing vs decreasing functions

INCREASING FUNCTIONS


DECREASING FUNCTIONS


- We specifically will talking about closed INTERVALS where functions increase or decrease.
- (Please note that functions can NOT be considered increasing or decreasing at a point itself rather we look at whether an interval is increasing or decreasing)
- Some textbooks look at this issue differently. Some textbooks use open intervals for increase and decrease - AP uses closed intervals

A function, $f(x)$, is said to be increasing on the closed interval [c, d] if all $f\left(x_{1}\right)<f\left(x_{2}\right)$ when the order of our $x$ terms satisfies $c<x_{1}<x_{2}<d$


A function, $f(x)$, is said to be decreasing on the closed interval [c, d] if all $f\left(x_{1}\right)>f\left(x_{2}\right)$ when the order of our x terms satisfies $\mathrm{c}<\mathrm{x}_{1}<\mathrm{x}_{2}<\mathrm{d}$


## Monotonicity

We say that $f(x)$ is monotonic on $(a, b)$ if it is either always increasing or always decreasing on $(a, b)$.

Ex \#4: State intervals of increase and intervals of decrease for the following graph.
NOTE: Unless the original graph has open endpoints, our intervals of increase and decrease are most accurately defined using CLOSED brackets. Unfortunately, many discrepancies between textbooks exist on this point. The max or min value that divides the areas of increase and decrease are included in both the increasing and decreasing INTERVALS (note that points themselves are not increasing or decreasing, only intervals). That max/min point is higher or lower than both the points next
 to it.

Ex \#3: Find where each of the following function is positive or negative. Give answers in interval notation.



Let $f$ be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$.

1. If $f^{\prime}(x)>0$ for all $x$ in $(a, b)$, then $f$ is increasing on $[a, b]$.
2. If $f^{\prime}(x)<0$ for all $x$ in $(a, b)$, then $f$ is decreasing on $[a, b]$.
3. If $f^{\prime}(x)=0$ for all $x$ in $(a, b)$, then $f$ is constant on $[a, b]$.


Statements 1 - 3 all allow for a finite (countable) number of points within the interval $[a, b]$ where
$f^{\prime}(x)=0$ (explanation below)

Note: The above definition includes a small caveat to take note of (in the right box above). Here it is defined:

Suppose $f(x)$ is a function that is continuous on $[a, b]$

- If $f^{\prime}(x)>0$ for all $x \in(c, d)$, with the exception of a finite number of points at which $f^{\prime}(x)=0$ or $f^{\prime}(x)$ DNA, then $f(x)$ is INCREASING on [c, d]. The finite points within the interval where $f^{\prime}(x)=0$ or $f^{\prime}(x)$ DNA will just be included within the increasing interval because technically the points to the left will be lower and the point on the right will be higher than the point itself, thus it will still be increasing over that interval. An example is the graph of $y=x^{3}$
- If $f^{\prime}(x)<0$ for all $x \in(c, d)$, with the exception of a finite number of points at which $f^{\prime}(x)=0$ or $f^{\prime}(x)$ DNA, then $f(x)$ is DECREASING on [ $c, d$ ]. The finite points within the interval where $f^{\prime}(x)=0$ or $f^{\prime}(x)$ DNA will just be included within the increasing interval because technically the points to the left will be higher and the point on the right will be lower than the point itself, thus it will still be decreasing over that interval. An example is the graph of $y=-x^{3}$

In general, we say that the derivative of a function will be positive for all values of $x$ in intervals where the graph is increasing and the derivative of a function will be negative for all values of $x$ in intervals where the graph is decreasing.

## STEPS TO FINDING INCREASING/DECREASING INTERVALS OF A FUNCTION f(x)

1. State appropriate intervals where $x$ is continuous and differentiable - this will include finding domain of $f(x)$
2. Find $f^{\prime}(x)$ in factored form (no negative exponents)
3. Find all critical values within the domain of $f(x)$
4. Do sign analysis on $f^{\prime}(x)$. Label the sign analysis. The values you use to test for within the sign analysis will include the solutions and zeros of $f^{\prime}(\mathbf{x})$ as well as any holes or asymptotes existing of $\mathbf{f}(\mathbf{x})$
5. $f(x)$ will be increasing on the closed intervals where $f^{\prime}(x)$ is positive. The sentence you write will often look like the following (it may be different when given different initial information):
$f(x)$ is increasing on the interval __write the closed interval here_because $f^{\prime}(x)$ is $>0$ on that interval (remember that there may possibly be a finite number of places where $f^{\prime}(x)=0$ or where $f^{\prime}(x)$ DNE)
6. $f(x)$ will be decreasing on the closed intervals where $f^{\prime}(x)$ is negative. The sentence you write will often look like the following (it may be different when given different initial information):
7. $f(x)$ is decreasing on the interval _ write the closed interval here_because $f^{\prime}(x)$ is < 0 on that interval (rememher that there mav nossihlv he a finite number nf nlares where $f^{\prime}(x)=0$ nr where $f^{\prime}(x)$ nNF)

Ex \#1: Use sign analysis to find the closed intervals in which the function $f(x)=\frac{1}{3} x^{3}+x^{2}-3 x+1$ is increasing and decreasing. Find the coordinates of any relative extrema and state if they are a maximum or a minimum. (Note: It is very important that you not only provide algebraic justification for your answer but also that your answer explains in words what the algebra shows)

## THE FIRST DERIVATIVE TEST:

Let $f(x)$ be continuous on the interval ( $\mathrm{a}, \mathrm{b}$ ) in which c is the only critical number. If $f(x)$ is differentiable on the interval (except possibly at $c$ then $f(c)$ can be classified as a relative minimum, relative maximum or neither by using the following criteria:

1. If $f^{\prime}(x)$ is negative to the left of $c$ and positive to the right of $c$, then there is a relative minimum at $x=c$. This info can be determined by a given graph or table of $f^{\prime}(x)$ or by using sign analysis on the equation of $f^{\prime}(x)$.
2. If $f^{\prime}(x)$ is positive to the left of $c$ and negative to the right of $c$, then there is a relative maximum at $x=c$. This info can be determined by a given graph or table of $f^{\prime}(x)$ or by using sign analysis on the equation of $f^{\prime}(x)$.
3. If $f^{\prime}(x)$ has the same sign to the right and left of $c$, then $f(c)$ is neither a relative minimum or a relative maximum. This info can be determined by a given graph or table of $f^{\prime}(x)$ or by using sign analysis on the equation of $f^{\prime}(x)$.

Ex \#2: Use sign analysis to find the closed intervals in which the function $f(x)=3 x^{\frac{2}{3}}-6 x^{\frac{1}{3}}$ is increasing and decreasing. Find the coordinates of any relative extrema and state if they are a maximum or a minimum. AGAIN ALWAYS SHOW THE SIGN ANALYSIS AND AN EXPLANATION OF THE SIGN ANALYSIS FOR FULL MARKS.

Ex \#3: Use sign analysis to find the closed intervals in which the function $f(x)=\frac{x^{2}}{x-2}$ is increasing and decreasing. Find the coordinates of any relative extrema and state if they are a maximum or a minimum.

Ex \#4: find the intervals in which the function $f(x)=x \sqrt{x+3}$ is increasing and decreasing. Find the coordinates of any relative extrema and state if they are a maximum or a minimum. https://www.desmos.com/calculator/iv6pfklwao

Ex \#5: What are the values of x for which the function f defined by $f(x)=\left(x^{2}-3\right) e^{-x}$ is increasing?
https://www.desmos.com/calculator/t9zuojncjc
Ex \#6: $f(x)$ is continuous and differentiable for all real numbers. A graph of $f^{\prime}(x)$ is below. $f^{\prime}(x)$ has zeroes at $\mathrm{x}=-3,1$, and 4. $f^{\prime}(x)$ has a relative maximum at $x=-1.5$ and $x=4$, as well as a relative minimum at $x=2$.
a) When is $f^{\prime}(x)=0$ ? Explain your reasoning in terms of what you see on the graph.

b) On what intervals is $f(x)$ increasing? Explain your reasoning in terms of the graph.
c) When does $f(x)$ have a relative minimum? Explain your reasoning using the graph.
d) When is $f(x)=0$ ? Explain your reasoning in terms of what you see on the graph.

### 5.3 Day 1: First Derivative Test and Critical Points

I GAN USE THE FIRST DERIVATIVE TEST TO UNDERSTAND HOW CRITGAL POINTS AFFECT THE GRAPH OF A FUNCTION VIDEO LINKS:
a) https://bit.ly/2T3816K up to time 14:30 b) https://bit.ly/2TbOBy7

## THE FIRST DERIVATIVE TEST FOR CRITICAL POINTS:

First Derivative Test for Critical Points - Let $f(x)$ be continuous on the interval $(a, b)$ in which $c$ is the only critical number. If $f(x)$ is differentiable on the interval (except possibly at $c$ ), then $f \subset$ can be classified as a relative minimum, relative maximum or neither, as follows:

1. If $f^{\prime}(x)$ is negative to the left of $c$ and positive to the right of $c$, then there is a relative minimum of $f(c)$ at $x=c$
2. If $f^{\prime}(x)$ is positive to the left of $c$ and negative to the right of $c$, then there is a relative maximum of $f(c)$ at $x=c$
3. If $f^{\prime}(x)$ has the same sign to the right and left of $c$, then $f(c)$ is neither a relative minimum or nor relative maximum and positive to the right of $c$, then there is a relative minimum of $f(c)$ at $x=c$

This also means that:

- If $f(x)$ changes from increasing to decreasing at $c$ then $f(c)$ is a LOCAL MAX
- If $f(x)$ changes from decreasing to increasing at $c$ then $f(c)$ is a LOCAL MIN


## Using the First Derivative Test (FDT) to Analyze Critical Points Analytically given $\boldsymbol{f}(\boldsymbol{x})$

1. State appropriate intervals where $x$ is continuous and differentiable - this will include finding domain of $f(x)$
2. Find $f^{\prime}(x)$
3. Find all critical values within the domain of $f(x)$
4. Do sign analysis on $f^{\prime}(x)$. Label the sign analysis. The values you use to test for within the sign analysis will include the solutions and zeros of $f^{\prime}(\mathbf{x})$ as well as any holes or asymptotes existing of $f(\mathbf{x})$
5. If your sign analysis shows that at $x=c, f^{\prime}(x)$ changes from positive to negative (or increasing to decreasing) then the critical point at $(\mathrm{c}, \mathrm{f}(\mathrm{c})$ ) is a local maximum. The sentence you write will often look like the following (it may be different when given different initial information):
$f(x)$ has a relative maximum of _write the value of $f(c)$ here_at $x=$ write the value of $c$ here because $f^{\prime}(x)$ changes from positive to negative at $x=$ write the value of $c$ here
6. If your sign analysis shows that at $x=c, f^{\prime}(x)$ changes from negative to positive (or decreasing to increasing) then the critical point at $(\mathrm{c}, \mathrm{f}(\mathrm{c})$ ) is a local minimum. The sentence you write will often look like the following (it may be different when given different initial information):
$f(x)$ has a relative minimum of _write the value of $f(c)$ here_ at $x=$ write the value of $c$ here because $f^{\prime}(x)$ changes from negative to positive to at $x=$ write the value of $c$ here

NOTE: if the question says to find the $x$-value of the local min or max, the answer is only $c$. If it says to find the local min or local max, substitute the critical points into $f(x)$ to find the $y$-value $f(c)$. Sometimes it will give the $\mathrm{max} / \mathrm{min}$ as a full ordered pair.

Ex \#1: Match each function with the graph of its derivative. (Note: The scales are not necessarily the same from one graph to the next)



Ex \#2: Shown below are the graphs of $f(x), f^{\prime}(x)$ and another function $g(x)$. Which is which?


Ex \#1: Find the critical points of $f(x)=x^{3}-27 x-20$ and determine if each is a local maximum, local minimum or neither.

Ex \#2: Analyze the critical points of $f(x)=\cos ^{2} x+\sin x$ on $[0, \pi]$. No Calculator!

Ex \#3: Analyze the critical points of $f(x)=\frac{1}{3} x^{3}-x^{2}+x$

Ex \#4: Analyze the critical points of $f(x)=\ln (x)-x$ on $(0, \infty)$. No Calculator!

## Using the First Derivative Test (FDT) to Analyze Critical Points graphically given $f^{\prime}(x)$

1. Identify all critical points by finding $x$-intercepts on the graph of $f^{\prime}(x)$ or places of discontinuity on $f^{\prime}(x)$ (this will be places where there are jumps or holes)
2. Identify where the graph of $f^{\prime}(x)$ has a positive position and where it has a negative position.
3. Where graph of $f^{\prime}(x)$ changes from a positive position to a negative position at some point $x=c$ (above to below the $x$-axis), then there is a local maximum at $f(c)$ - but we cannot find the value of $f(c)$
4. Where the graph of $f^{\prime}(x)$ changes from a negative position to a positive position at some $x=c$, then there is a local minimum at $f(c)$ - but we cannot find the value of $f(c)$
5. Write a concluding statement using proper language!

Ex \#4: Below is the graph of $f^{\prime}(x)$. Find the critical points of $f(x)$ and determine if each is a local minimum, local maximum or neither
a)

b)


d)


Ex \#6: Below is the graph of $f^{\prime}(x)$. Find the intervals where $f(x)$ is increasing or decreasing.

b)


Ex \#6: Given $f(x)=x^{2} \sqrt{5-x}$, list all critical numbers, open intervals of increase and decrease and relative extrema. Sketch a graph and label all critical points. (do not justify)

### 5.3 Day 2: Connecting The $2^{\text {nd }}$ Derivative to Concavity \& Inflection Points

 Calculus
## CAN DETERMINE INTERVALS IN WHICH A CURVE IS CONGAVE UP OR DOWN AND FIND POINTS OF INFLECTION

## VIDEO LINKS:

a) https://bit.ly/2T3816K - after time 14:30 b) https://bit.ly/2zTZTOL

- We have already learned that the first derivative test tells us where a function $\mathrm{f}(\mathrm{x})$ is increasing $f^{\prime}(x)>0$ and decreasing $f^{\prime}(x)<0$. The first derivative test is also used to find relative extrema.
- In this lesson, we will determine what the second derivative tells us about $\mathrm{f}(\mathrm{x})$.


## REVIEW: Increasing vs Decreasing

A function $f(x)$ is INCREASING on an interval if all slopes of tangent lines are positive $\mathrm{f}^{\prime}(\mathrm{x})>0$ (except for a finite number of points where $\left.f^{\prime}(x)=0\right)$

- This graph is going "uphill"


A function $f(x)$ is DECREASING on an interval if all slopes of tangent lines are negative $\mathrm{f}^{\prime}(\mathrm{x})<0$ (except for a finite number of points where $f^{\prime}(x)=0$ )

- This graph is going "downhill"



## REVIEW: CONCAVE UP

- Concave up intervals are shaped like a valley
- Concavity describes how the slopes of the tangent lines (which are the first derivative) changes as we look at the tangent lines from the left to the tangent lines to the right.
- This means we are looking at the rate of change of the first derivative. A rate of change IS a derivative - this means we are taking the derivative of the first derivative, which is the SECOND derivative.


As you move from left to right, are the slopes becoming larger and increasing (more positive) or smaller and decreasing (more negative)? $\qquad$ . Since this answer is describing the SECOND derivative, this means that a concave up interval will have a $\qquad$ second derivative.

## Let $f(x)$ be a differentiable function on an open interval $(a, b)$, then $f(x)$ is CONCAVE UP if:

- The second derivative exists (which also means that the first derivative also exists
- The first derivative is always increasing (if the slopes of the tangent lines are moving from small to big).
- The graph of $f(x)$ is always ABOVE the tangent lines
- Concave up on (a, b) if $f^{\prime}(x)$ is increasing on ( $a, b$ )


Slopes of tan lines increase

- $\therefore$ Concave up are open intervals where $\mathrm{f}^{\prime \prime}(\mathrm{x})>0$


## REVIEW: CONCAVE DOWN

- Concave down intervals are shaped like a hill

As you move from left to right, are the slopes becoming larger and increasing (more positive) or smaller and decreasing (more negative)? $\qquad$ Since this answer is describing the SECOND derivative, this means that a concave up interval will have a ___(negative or positive) second
 derivative.

## Let $f(x)$ be a differentiable function on an open interval $(a, b)$, then $f(x)$ is CONCAVE DOWN if:

- The second derivative exists (which also means that the first derivative also exists
- The first derivative is always decreasing (if the slopes of the tangent lines are moving from big to small).
- The graph of $f(x)$ is always BELOW the tangent lines

- Concave up on ( $\mathrm{a}, \mathrm{b}$ ) if $f^{\prime}(x)$ is increasing on ( $a, b$ )
- $\quad \therefore$ Concave down are open intervals where $\mathrm{f}^{\prime \prime}(\mathrm{x})<0$


## Test for Concavity

Let $f$ be a function whose second derivative exists on an open interval $I$.

1) If $f^{\prime \prime}(x)>0$ for all x in $I$, then the graph of $f$ is concave upward in $I$.
2) If $f^{\prime \prime}(x)<0$ for all x in $I$, then the graph of $f$ is concave downward in $I$.

Ex \#1: State the open intervals on which the function is concave up and concave down.


## Definition of Point of Inflection

If the graph of a continuous function possesses a tangent line at a point where its concavity changes from upward to downward (or vice versa), then the point is a point of inflection.




The concavity of $f$ changes at a point of inflection.
The graph crosses its tangent line at a point of inflection.

## MORE ABOUT POINTS OF INFLECTION:

- They are points where the concavity of a function changes

Concave up to Concave Down Concave Down to Concave Up


- At the point of inflection, $f^{\prime \prime}(x)=0$ or $f^{\prime \prime}(x)$ does not exist (BUT THE TANGENT LINE $f^{\prime}(x)$ does exist) Unfortunately not all textbooks agree with this but AP follows this guideline.
- WORDS OF WARNING: In order for the value of $x$ where $f^{\prime \prime}(x)=0$ or $f^{\prime \prime}(x)$ does not exist to be called a Point of Inflection, the point must be included as a point in the original domain of $f(x)$ and the concavity must be different on either side of the point(the concavity must change) You need to do a sign analysis on $f^{\prime \prime}(x)$ to verify that the concavity changes! Do NOT include the Inflection Point (IP) in the interval of concavity!

Theorem: Let $f(x)$ be a differentiable function on an open interval $(a, b)$. Then $f(x)$ is

- Concave up on (a, b) if $f^{\prime \prime}(x)$ is positive for all x in (a, b)
- Concave down on (a, b) if $f^{\prime \prime}(x)$ is negative for all x in (a, b)


Test for inflection points. Assume that $f^{\prime \prime}(x)$ exists (which implies that $\mathrm{f}^{\text {' }}(\mathrm{x})$ also exists) for all $x \in(a, b)$ and let $c \in$ $(a, b)$. If $f^{\prime \prime}(c)=0$ or $f^{\prime \prime}(c)$ does not exist AND $f^{\prime \prime}(x)$ changes sign at $\mathrm{x}=\mathrm{c}$, then $f(x)$ has a point of inflection at $\mathrm{x}=\mathrm{c}$.

USING THE SECOND DERIVATIVE TO FIND INFLECTION POINTS AND INTERVALS OF CONCAVITY OF f(x)

1. State appropriate intervals where $x$ is continuous and differentiable - this will include finding domain of $f(x)$
2. Find $f^{\prime}(x)$ and state it's domain
3. Find $f$ " $(x)$ and leave it in factored form.
4. Find all x -values where $f^{\prime \prime}(x)$ is zero or undefined. (these are possible inflection points - PIPs)
5. Do Sign analysis for $f$ " $(x)$ (and label it $f$ " $(x)$ ) on the PIP values for $x$. If there are any points where $f(x)$ or $f$ ' $(x)$ are undefined, include these $x$ values in your sign analysis.
6. The ACTUAL Points of Inflection are only the Possible Points of Inflection (PIPS) where $\mathrm{f}^{\prime \prime}(\mathrm{x})$ changes from positive to negative or negative to positive. If you are asked to state the inflection points, . The sentence you write will often look like the following (it may be different when given different initial information):
$f(x)$ has points of inflection at write all ordered pairs ( $c, f(c)$ ) of inflection points here because
$f^{\prime \prime}(x)$ changes signs at write all the $x$ values of all the inflection points using $x=c, x=c_{1}$ etc
7. Open intervals where $f^{\prime \prime}(x)$ is positive indicates that $f$ is concave up on that open interval. The sentence you write will often look like the following (it may be different when given different initial information):
$f(x)$ is concave up on write the open intervals here_because $f^{\prime \prime}(x)>0$ on this/these intervals
8. Open intervals where $f^{\prime \prime}(\mathrm{x})$ is negative indicates that $f$ is concave down on that open interval. The sentence you write will often look like the following (it may be different when given different initial information):
$f(x)$ is concave down on . write the open intervals here because $f^{\prime \prime}(x)<0$ on this/these intervals

Ex \#2: Find the points of inflection of $f(x)=3 x^{5}-5 x^{4}+1$ and determine the intervals where $f(x)$ is concave up and concave down. PLEASE ASSUME THAT ALL QUESTIONS I ASK REQUIRE JUSTIFICATION.

Ex \#3: Find the points of inflection of $f(x)=\frac{24}{x^{2}+12}$ and determine the open intervals where $f(x)$ is concave up and concave down.

Ex \#4: Given $f(x)=3 x^{\frac{2}{3}}-6 x^{\frac{1}{3}}$ find the points of inflection and the open intervals where it is concave up and down. https://www.desmos.com/calculator/5g4qwp0cfo

## Finding intervals of concavity and inflection points given the graph of $\boldsymbol{f}^{\prime}(\boldsymbol{x})$

- If $f^{\prime}(x)$ is increasing on $(a, b), f(x)$ is concave up on $(a, b)-f(x)$ is concave up on intervals where $f^{\prime}(x)$ has positive slope
- If $f^{\prime}(x)$ is decreasing on $(\mathrm{a}, \mathrm{b}), \mathrm{f}(\mathrm{x})$ is concave down on $(\mathrm{a}, \mathrm{b})-f(x)$ is concave down on intervals where $f^{\prime}(x)$ has negative slope
- Inflection points occur on the graph of $f^{\prime}(x)$ at local extrema - inflection points of $f(x)$ show up as max/mins on the graph off' $(x)$

Ex \#5: given the graph below of $f^{\prime}(x)$, find inflection points and intervals of concavity.
a)

b)


## Finding intervals of concavity and inflection points given the graph of $\boldsymbol{f}^{\prime \prime}(\boldsymbol{x})$

- If $f^{\prime \prime}(x)$ is positive (above the $x$-axis), $f(x)$ is concave up
- If $f^{\prime \prime}(x)$ is negative (below the $x$-axis), $f(x)$ is concave down
- Inflection points occur on the graph of $f^{\prime \prime}(x)$ at $x$-intercepts where the graph changes from above to below or below to above the x -axis

Ex \#6: given the graph below of $f^{\prime \prime}(x)$, find the inflection points and intervals of concavity.


Ex \#7: The second derivative of the function $f$ is given by $f^{\prime \prime}(x)=x(x-a)(x-b)^{2}$. The graph of $f^{\prime \prime}(x)$ is shown to the right. For what values of $x$ does the graph of $f^{\prime}(x)$ have a relative maximum?
A. $j$ and $k$ only
B. $a$ and $b$ only
C. a only
D. 0 only
E. $a$ and 0 only


Ex \#8: A table of function values for a twice differentiable function, $f(x)$, is pictured to the right. Which of the following statements is/are true if $f(x)$ has only one zero on the $-3 \leq x \leq 3$ ?
I. $f^{\prime}(x)<0$ on the interval $-3<x<3$.
II. $f(x)$ has a zero between $x=1$ and $x=3$.
III. $f^{\prime \prime}(x)>0$ on the interval $-3<x<3$.

| $x$ | $f(x)$ |
| :---: | :---: |
| -3 | 10 |
| -1 | 8 |
| 1 | 2 |
| 3 | -13 |

A. I only
B. I and II only
C. III only
D. II and III only E. I, II and III

## Ex \＃9：Calculator Active Question

The function $f^{\prime}(x)=\cos (\ln x)$ is the first derivative of a twice differentiable function，$f(x)$ ．
a．On the interval $0<x<10$ ，find the $x$－value（s）where $f(x)$ has a relative maximum．Justify your answer．
b．On the interval $0<x<10$ ，find the $x$－value（s）where $f(x)$ has a relative minimum．Justify your answer．
c．On the interval $0<x<10$ ，find the $x$－value（s）where $f(x)$ has a point of inflection．Justify your answer．

On the interval $0<x<10$ ，how many relative minimums does the graph of $g(x)$ have if $g^{\prime}(x)=\frac{\sin x}{x+2}$ ？
A． 0
B． 1
C． 2
D． 3
E． 4

### 5.3 Day 3: The Second Derivative Test to Find Extrema

GAN APPLY THE FIRST AND SECOND DERIVATIVES TO DETERMINE RELATIVE AND ABSOLUTE EXTREMA
VIDEO LINKS: a) https://bit.ly/2DDgOJF
The second derivative: In some situations, we can use both the first and second derivative to determine local maximum and minimum values of a function $f(x)$.

Look at the graphs of $f(x)$ on the right. What do we know about $f^{\prime}(c)$ and $f^{\prime \prime}(c)$ ?

In Figure 8, when $\mathrm{x}=\mathrm{c}$ :

- The slope of the tangent line is 0 , therefore $\mathbf{f}^{\prime}(\mathbf{c})=\mathbf{0}$. This indicates that $\mathbf{f}(\mathbf{c})$ is a relative max or min value.
- The slope of the tangent lines are decreasing, therefore $\mathbf{f l \prime}^{\prime \prime}(c)<0$. This indicates that the curve is concave down, which means that $f(c)$ must be a relative maximum value.

In Figure 9, when $\mathrm{x}=\mathrm{c}$ :

- The slope of the tangent line is 0 , therefore $f^{\prime}(c)=0$. This indicates that $f(c)$ is a relative max or min value.
- The slope of the tangent lines are increasing, therefore $\mathbf{f}^{\prime \prime}(\mathrm{c}) \boldsymbol{>} \mathbf{0}$. This indicates that the curve is concave up, which means that $f(c)$ must be a relative minimum value.


Figure 9

## THE SECOND DERIVATIVE TEST:

Suppose that $f(x)$ is a continuous function in an interval containing $x=c$ where $f^{\prime}(c)=0$ (which means that $c$ is a critical number) and the second derivative of $f(x)$ exists on an open interval containing $c$.

1. If $f^{\prime \prime}(c)>0(f(x)$ would be concave up) then $f$ has a local minimum of $f(c)$ at $c$.
2. If $f^{\prime \prime}(c)<0(f(x)$ would be concave down) then $f$ has a local maximum of $f(c)$ at $c$.
3. CAUTION: If $f^{\prime \prime}(c)=0$ or $f^{\prime \prime}(c)$ DNE, the second derivative test fails. You can then use the first derivative test (which never fails) to locate a relative extremum. You would then need to do a sign analysis to determine whether or not $f(c)$ is a relative maximum or minimum or neither.

Ex \#1: Use the second derivative test to find the local max and min values of $f(x)=x^{3}-12 x+5$.
https://www.desmos.com/calculator/nfpmkc6dkk

Ex \#2: Use the second derivative test to find the relative extrema of $f(x)=\frac{1}{4} x^{4}-x^{3}$.

WORD of WARNLNG: : If $\mathrm{f}^{\prime \prime}(\mathrm{c})=0$ or $\mathrm{f}^{\prime \prime}(\mathrm{c})$ DNE, the second derivative test fails. You can then use the first derivative test (which never fails) to locate a relative extrema. You would then need to do a sign analysis to determine whether or not $f(c)$ is a relative maximum or minimum or neither.

Ex \#3: A graph of $f^{\prime}(x)$ is given below. (ignore the $y$ values other than zero)
a) On what intervals is $f$ increasing
b) On what intervals is $f$ concave up?
c) At which $x$-values does $f$ have a local extrema?
d) What are the $x$-values of the inflection points of the graph of $f$ ?
e) Sketch a possible graph of $f$ on $[-4,3]$



Ex \#4: Let $f^{\prime}(x)=4 x^{3}-12 x^{2}$
a) Identify where the extrema of $f$ occur
b) Find the intervals on which $f$ is increasing and the intervals on which $f$ is decreasing
c) Find where the graph if $f$ is concave up and where it is concave down.
d) Sketch a possible graph for $f$.

The $1^{\text {st }}$ derivative test should be your "go to" test when searching for relative extrema of a function. It is always conclusive and involves the least amount of work.

However, there are circumstances that will require you to use the $2^{\text {nd }}$ derivative test.
The first derivative test depends on your creating a sign chart and plugging in $x$-values to determine if a function is increasing or decreasing on specific intervals. This means that the function you are using must be explicit (in terms of x ). However, some functions and their derivatives are implicit (in terms of $x$ and $y$ ). For these cases, we are unable to use a sign chart because we must use both $x$ and $y$ values. This leads us to use the $2^{\text {nd }}$ derivative test.

## Ex \#5:

Consider the curve given by $x^{2}+4 y^{2}=7+3 x y$.
(a) Show that $\frac{d y}{d x}=\frac{3 y-2 x}{8 y-3 x}$.
(b) Show that there is a point $P$ with $x$-coordinate 3 at which the line tangent to the curve at $P$ is horizontal. Find the $y$-coordinate of $P$.
(c) Find the value of $\frac{d^{2} y}{d x^{2}}$ at the point $P$ found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point $P$ ? Justify your answer.

Ex \#6: Let the following be the graph of $f^{\prime}(x)$.

Find x -values of the critical points of $f$.

Find intervals of increasing/decreasing of $f(x)$ and justify.


Find the x -values of the local minimum(s) and local Maximum(s) of $f(x)$ and justify.

Find the intervals of concavity of $f(x)$ and justify.

Find the $x$-values of inflection points of $f(x)$ and justify.

## Ex \#7: Connecting the Graphs of $f(x), f^{\prime}(x)$, and $f^{\prime \prime}(x)$

To the right is the graph of a function, $F(x)$. State all of the conclusions that you can state about the graphs of $F^{\prime}(x)$ and $F^{\prime \prime}(x)$. Justify each of your conclusions.

Graph of $F(x)$


| Conclusions about $F^{\prime}(x)$ | Conclusions about $F^{\prime \prime}(x)$ |
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Based on the relationships between a function and its first and second derivative, complete the following statements.

1. $f(x)$ is increasing
2. $f(x)$ is decreasing
3. $f(x)$ has a relative maximum or minimum
4. $f(x)$ has a point of inflection
5. $f(x)$ is concave up
6. $f(x)$ is concave down
7. $f(x)$ has a point of inflection
8. $f^{\prime}(x)$ is increasing
9. $f^{\prime}(x)$ is decreasing
10. $f^{\prime}(x)$ has a relative maximum or minimum
11. $f^{\prime}(x)$ has a point of inflection
12. $f^{\prime}(x)$ changes from negative to positive
13. $f^{\prime}(x)$ changes from positive to negative
14. $f^{\prime}(x)$ has a relative maximum or minimum
15. $f^{\prime \prime}(x)$ changes from positive to negative
16. $f^{\prime \prime}(x)$ changes from negative to positive

$$
\begin{aligned}
& \leftrightarrow \quad f^{\prime}(x) \\
& \leftrightarrow \quad f^{\prime}(x) \\
& \leftrightarrow \quad f^{\prime}(x) \\
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& \leftrightarrow \quad f^{\prime}(x) \\
& \leftrightarrow \quad f^{\prime}(x)
\end{aligned}
$$




One of the more challenging situations is being able to distinguish between which existence theorem a question is referring to. The following two questions show a side by side contrast between the Intermediate Value Theorem and the Mean Value Theorem.

## Quick Review:

IVT says:
MVT says:

| Intermediate Value Theorem (IVT) |  |  |  |  | Mean Value Theorem (MVT) |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Conditions: |  |  |  |  | Conditions: |  |  |  |  |
| What to look for: |  |  |  |  | What to look for: |  |  |  |  |
| Example: |  |  |  |  | Example: |  |  |  |  |
| $t$ (min) | 0 | 1 | 5 | 9 | $t$ (min) | 0 | 1 | 5 | 9 |
| $v(t)(f t / \mathrm{min})$ | 5 | 2 | -7 | -4 | $v(t)(f t / \mathrm{min})$ | 5 | 2 | -7 | -4 |
| A bug is walking along the $x$-axis. The bug's velocity is continuous and differentiable and selected values are given in the table above. Is there a time $c$, for $1<c<5$, such that <br> $v(c)=0$ ? Give a reason for your answer. |  |  |  |  | A bug is walking along the $x$-axis. The bug's velocity is continuous and differentiable and selected values are given in the table above. Is there a time $c$, for $0<c<9$, such that <br> $a(c)=-1$ ? Give a reason for your answer. |  |  |  |  |
| Work: |  |  |  |  | Work: |  |  |  |  |
| Statement: |  |  |  |  | Statement: <br> Since $\qquad$ is $\qquad$ on $\qquad$ and $\qquad$ on $\qquad$ , the $\qquad$ guarantees that... |  |  |  |  |
| Since $\qquad$ is $\qquad$ on $\qquad$ <br> the $\qquad$ guarantees that... |  |  |  |  |  |  |  |  |  |


[^0]:    Rolle's Theorem Applies
    
    
    

