GAN USE LINEAR APPROXIMATION TO ESTIMATE THE VALUE OF A FUNCTION NEAR A POINT OF tangency \& find the differential of a function

## VIDEO LINKS:

a) https://bit.ly/2Pcox2B
\& https://bit.ly/2SvxM08
b) https://bit.Iy/2QwGeP7
c) https://bit.ly/2LMFSiB
d) https://bit.ly/2CGo9Gx (differentials)

EX \#1: Write an equation of the line tangent to the curve of $f(x)=\sqrt{x}$ where $\mathrm{x}=16$. Leave your answer in point slope form. Sketch both.

## https://www.desmos.com/calculator/ztkj2uicjc



## DEVELOPMENT: Use your work in ex\#1 to Develop a method using linear approximation to approximate the value of $\sqrt{16.3}$

- The tangent line to a curve is a good approximation to the curve itself near the point of tangency - if you were to zoom in on the graph you would see what appears to be two parallel lines very close to each other.
- Due to this zoomed in quality, we say that differentiable curves are LOCALLY LINEAR.
- On the following graph, we can see that the point of tangency is at $(a, f(a))$.
- Using a highlighter, please mark the point on the black curved graph of $f(x)$ that corresponds to the given value of $x$

- Using a different highlighter, please mark the point on the tangent line that corresponds to the given value of $x$
- The $y$ value of the point on the line of tangency is a good approximation for the $y$ value of the graph of $f(x)$ (in our case it will be a bit higher)
- When we want to find the height, $f(x)$ at a point $x$, we can approximate its value by finding the value of $f(x)$ that exists on the tangent line. This is called LINEAR APPROXMATION or LINEARIZATION.
- Linear approximation is helpful when the calculation to find the actual y value of the function at x is very cumbersome (particularly if you don't have a calculator) or when we don't know the equation for $f(x)$ but have information about the tangent line at a point. Historically this method has been used to approximate the value of square roots.


## LINEARIZATION

If $f$ is differentiable at $x=a$, then the equation of the tangent line $L(x)$ can be developed as followed: $y-y_{1}=m\left(x-x_{1}\right)$, where our point is $(\mathrm{a}, \mathrm{f}(\mathrm{a}))$ and our slope at that point is $\mathrm{f}^{\prime}(\mathrm{a})$
$y-f(a)=f^{\prime}(a)(x-a)$
$y=f^{\prime}(a)(x-a)+f(a)$
Let the name of our tangent line be called $L(x)$
$\therefore L(x)=f^{\prime}(a)(x-a)+f(a)$


EX \#2: Write an equation of the tangent line to $f(x)=x^{3}$ at $(2,8)$. Use the tangent line to approximate the values for $f(1.9)$ and $f(2.01)$. (NO CALCULATOR)

EX \#3: Find the linearization of $f(x)=\cos x$ at $x=\frac{\pi}{2}$ and use it to approximate $\cos 1.75$ (NO CALCULATOR).

EX \#4: Use a linearization to approximate $\sqrt{4.1}$

EX \#5: Use the linearization formula to calculate the margin of error for the approximation answer you found in ex 4.

## Over-approximation or Under-approximation

To determine whether our tangent line approximation is an over or under approximation, we need to determine of the tangent line is above or below that part of the curve.

- If the curve is upward facing at the point of tangency, our tangent line will be below the curve so we will have an under approximation
- If the curve is downward facing at the point of tangency, our tangent line will be above the curve so we will have an over approximation.
QUESTION: How does this information relate to Ex\#4 and Ex\#5 above?


## DIFFERENTIALS

- We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes as a result of a small change in the input (as in the margin of error above). This leads us to the topic of DIFFERENTIALS.
- Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.
- When we first looked at derivatives, we used the Leibniz notation $\frac{d y}{d x}$ to represent the derivative of $y$ with respect to $x$. Although we used the expressions $d y$ and $d x$ in this notation, they did not have meaning on their own - it was just used as part of $f^{\prime}(x)=\frac{d y}{d x}$ or $f^{\prime}(x)=\frac{\Delta y}{\Delta x}$

DIFFERENTIALS give we see a meaning to the expressions $d y$ and $d x$.
Suppose $y=f(x)$ is a differentiable function. Let $d x$ be an independent variable that can be assigned any nonzero real number, and define the dependent variable $d y$ by $d y=f^{\prime}(x) d x$ or $\Delta y=f^{\prime}(x) \Delta x$ This equation is called the DIFFERENTIAL FORM of the equation $f^{\prime}(x)=\frac{d y}{d x}$ It is important to notice that $d y$ is a function of both $x$ and $d x$. The expressions $d y$ and $d x$ are called differentials.

EX \#6: a) If $y=f(x)$ and given $f(x)=x^{2}+2 x$, find $d y$ and evaluate when $x=3$ and $d x=0.1$
b) Given $y=\sin 3 x$, find $d y$ and evaluate when $x=\pi, d x=-0.02$
5.4 Optimization - CALCULUS 30 L TOPIC - THIS TOPIC IS TAUGHT IN AP AS FINDING ABSOLUTE MAXIMUM OR MINIMIM (OFTEN ALONG WITH THE THREE BIG THEOREMS)
Optimization in AP Calculus (as well as in Calculus 30L) is taught only in terms of finding Absolute Max or Min and is usually assessed in conjunction with the three big theorems. Please see the note at the top of the next section for the regular Calculus 30 perspective. The Calculus 30 section will be taught AFTER the AP exam unless time permits otherwise.

1) The total number of gallons of water in a tank at time $t$ is modeled by the expression

$$
A(t)=30+8 t-\frac{2}{3}(t+1)^{\frac{3}{2}}
$$

gallons, where $t$ is measured in minutes. At what time $t$, for $0 \leq t \leq 120$, is the amount of water in the tank a maximum? Show the Calculus analysis that leads to your conclusion.
2) The number of gallons, $P(t)$, of a pollutant in a lake is modeled by

$$
P(t)=\frac{2}{9} t^{\frac{3}{2}}-2 t+50
$$

gallons, where $t$ is measured in days. What is the minimum number of gallons of pollutant? Justify your answer with Calculus.
3) For $0 \leq t \leq 31$, the number of mosquitoes on Tropical Island at time $t$ days is modeled by

$$
M(t)=20 \sqrt{t} \sin \left(\frac{t}{5}\right)+1000
$$

mosquitoes. To the nearest whole number, what is the maximum number of mosquitoes for $0 \leq t \leq 31$ ? Show the Calculus analysis that leads to your conclusion.
(This question is Calculator Active)
4) The number of people in an amusement park on a given day is modeled by

$$
H(t)=-80 x^{2}+2560 x-16530
$$

people, where time $t$ is measured in hours after midnight. This function is valid for $9 \leq t \leq 23$, the hours during which the park is open. At what time $t$, for $9 \leq t \leq 23$, does the model predict that the number of people in the park is a maximum? Show the Calculus analysis that leads to your conclusion.
5.4 Optimization - CALCULUS 30 TOPIC (MAY BE TAUGHT AFTER THE AP EXAM UNLESS TIME PERMITS OTHERWISE

## GAN IDENTIFY, SET UP AND SOLVE OPTIMIZATION PROBLEMS

VIDEO LINKS:
a) https://bit.ly/2Qb55bV
b) https://bit.ly/2FXZKjl
c) https://bit.ly/2QFbATO

The Calculus 30 perspective on Optimization is in line with how Optimization was taught in PreCalculus courses where there is a word problem followed by an algebraic setup of equations. In PreCalculus, algebra was used to find the Optimum value. In Calculus, we will use Calculus to find the Optimum value.
In this section we work on word problems where we try to find the maximum or minimum value of a variable. The solution is referred to as the optimum solution.

## TIPS FOR SOLVING OPTOMIZATION PROBLEMS:

## 1. UNDERSTAND THE PROBLEM:

Optimization problems often appear in the form of a word problem. It is most important that you read the problem over carefully until you have a clear idea as to what it is you are being asked to maximize or minimize. You may want to experiment with some hypothetical "answers" to bring clarification to the problem and assist you in creating the appropriate function.
2. DRAW A DIAGRAM:

Many optimization problems require a diagram. Label your diagram carefully with all the given dimensions. Use one or more variables to label dimensions that you do not know. Using meaningful variables, such as $h$ for height and $r$ for radius is helpful. Write let statements and state the domain for variables.

## 3. DETERMINE THE FUNCTION:

Set up a function that gives an expression for the quantity you are trying to optimize in terms of the constants and variables you established in step 2. Initially this function may be in terms of more than one variable. Read your problem carefully and re-examine your diagram to see how the variables are interrelated. The Pythagorean Theorem can also be helpful! You must write your optimization function in terms of only one variable before you can proceed. Use function notation

## 4. FIND THE DOMAIN OF THE FUNCTION:

Read the question carefully and determine the domain. It will be relevant in step 5 .
5. DETERMINE THE GLOBAL MAXIMUM OR MINIMUM:

Assuming the interval is closed, follow these steps:
a) Find the derivative of the function
b) Determine the critical numbers
c) Do a sign analysis of the derivative on the domain of the original function. Establish and justify any relative extrema
d) Evaluate the function at each endpoint of the interval
e) Determine your global max or min from the points in steps c) and d) above.

- If your domain is an open interval, you may need to use limits to determine what happens to the optimization function as the variable approaches the boundaries of the open interval.


## 6. BE SURE YOU HAVE ANSWERED THE ACTUAL QUESTION!

Ex \#1: What positive number, when added to its reciprocal, gives the least possible sum? What is that sum?

Ex \#2: Two nonnegative numbers have a sum of 10. Find these numbers if the sum of their cubes is to be a minimum.

Ex \#3: A farmer has 1000 m of fencing and wants to enclose a rectangular pasture bordering a straight, swift river. The farmer thinks that the cattle will not wander into the river and thus a fence will not be needed along that side. Find the dimensions that will enclose the maximum area, and find that area.

Ex \#4: A rectangular sheet of paper of dimensions $18 \mathrm{~cm} \times 25 \mathrm{~cm}$ is to be made into an open box by cutting out equal squares from the corners and bending up the flaps. Find the length of the cut out squares so that the volume of the box will be a maximum. What will the maximum volume be?

Ex \#5: A rectangular piece of metal 60 cm wide and several hundred metres long is to be folded along the centre of its length in order to make a long trough for irrigation. What should be the width of the trough at the top in order to maximize its carrying capacity? You may assume that the carrying capacity will be maximized if the area of the triangular cross section of the trough is maximized.

Ex \#6: A can of pop is to hold $400 \mathrm{ml}\left(400 \mathrm{~cm}^{3}\right)$. Find the radius and height of the can if the amount of metal used is to be minimized. Ignore the thickness of the metal.

Ex \#7: Find the dimensions of the cylinder of largest volume that can be inscribed in a sphere of radius 10 cm.


Ex \#8: A rectangle is to be inscribed under one arch of the sine curve. What is the largest area the rectangle can have, and what dimensions give that area? CALCULATOR ACTIVE

$[0, \pi]$ by $[-0.5,1.5]$

### 5.6 Related Rates Day 1 (Both 30 and 30L)

CAN IDENTIFYAND SOLVERELATED RATES PROBLEMS USING IMPLICIT DIFFERENTIATION VIDEO LINKS:
a) https://bit.ly/2BRjalG
b) https://bit.ly/2DYUSbc
c) https://bit.ly/2zKQOTZ

EX \#1: Implicitly differentiate the following formulas with respect to time. State what each rate in the differential equation represents

| 1. $A=4 \pi r^{2}$ |  |
| :---: | :---: |
| Surface Area of a Sphere |  |
| 2. $V=\frac{4}{3} \pi r^{3}$ |  |
| Volume of a Sphere |  |
| 3. $a=\sqrt{c^{2}-b^{2}}$, where $c$ is is constant |  |
| 4. $V=\pi r^{2} h$, where $r$ is constant |  |
| Volume of a Cylinder |  |
| 5. $\cos \theta=\frac{x}{15}$ |  |
| 6. $V=\frac{1}{3} \pi r^{2} h$ |  |
| Volume of a Cone |  |

## .TIPS FOR SOLVING RELATED RATES PROBLEMS DEALING WITH RIGHT TRIANGLES:

1. UNDERSTAND THE PROBLEM- Read the question at LEAST twice!
2. DRAW A DIAGRAM: Draw the triangle representing the problem.

- Layer 1: Label all side with $\mathrm{x}, \mathrm{y}$ and z (hypotenuse).
- Layer 2: Write the length (distance) of all known sides.
- Layer 3: Write all known rates (which will be velocities) occurring at each side. Label them $\frac{d x}{d t}, \frac{d y}{d t}$ and $\frac{d z}{d t}$.
- Note : If the length of the side is decreasing, the velocity will be negative.

It is possible that one of the velocities may be 0 if there is no movement.
3. DETERMINE ANY MISSING DISTANCES:

- Using the Pythagorean Theorem, determine the missing values of $\mathrm{x}, \mathrm{y}$ or z .
- Write these in your diagram. You may need to create a new one.

4. IMPLICITLY DIFFERENTIATE THE PREVIOUS EQUATION (PYTHAGOREAN THEOREM) relative to time.
5. SIMPLIFY AND SUBSTITUTE ANY KNOWN VALUES INTO THE DIFFERENTATIATED EQUATION:

- When substituting, write the units of measurement.
- You should now have a single unknown variable or rate, which is the one you are looking for.

6. ANSWER THE QUESTION IN A SENTENCE: Include units of measurement.

## TIPS FOR SOLVING RELATED RATES PROBLEMS (with area and volume):

## 1. UNDERSTAND THE PROBLEM

2. CREATE THE WHENEVER DIAGRAM AND EQUATION (the diagram represents an object changing over time whenever just means your drawing represents a moment of that time period)

- Draw and label your diagram with all of the timeless facts:
- Variables, constants and rates of change.
- Write the equation that links these variables and constants. Possible formulas are provided on the next page.
- Identify what variable or rate you are trying to find.

3. A) IMPLICITLY DIFFERENTIATE THE PREVIOUS EQUATION RELATIVE TO TIME, then...

- Add your "when" information to your diagram.
- Substitute all known rates of change and all "when" information into your new equation.
- You should be left with only the one unknown, which should represent the answer to your question.

3. B) IF THE IMPLICIT DIFFERENTIATION WOULD LEAVE YOU WITH TOO MANY UNKNOWNS...

- Before differentiating, use the geometry of the situation to eliminate the unknown variables by substitution. (ex: ratio)
- Now, implicitly differentiate the equation with respect to $t$.
- Substitute all known rates of change and all "when" information into your new equation.
- You should be left with only the one unknown variable or rate, which should represent the answer to your question.

4. ISOLATE YOUR REMAINING UNKNOWN
5. ANSWER THE QUESTION WITH A FINAL STATEMENT

- Include units of measurement.
- In your statement, your velocities should be expressed at positives. Use the terms increase and decrease. (ie: The velocity is increasing at a rate of $5 \mathrm{~m} / \mathrm{s}$ or the velocity is decreasing at a rate of $5 \mathrm{~m} / \mathrm{s}$ - do not use negative signs when using the word decreasing!)

FORMULAS (These will be on the formula sheet for my test. Questions needing these formulas on the AP exam will have the formula included within the question.

Area: $\quad$ Triangle: $A=\frac{b h}{2}$
Perimeter: Square: $P=4 l$
Surface Area:
Closed rectangular box: $S=2 l w+2 l h+2 w h \quad$ Right circular cylinder (open at top and bottom): $S=2 \pi r h$ Sphere: $S=4 \pi r^{2} \quad$ Right circular cone with open base: $S=\pi r l$ where $l=$ slant height $=\sqrt{r^{2}+h^{2}}$
Volume: $\quad$ Prism or cylinder: $V=($ area of base $) \mathrm{h} \quad$ Sphere: $V=\frac{4}{3} \pi r^{3} \quad$ Cone: $V=\frac{1}{3} \pi r^{2} h$

EX \#2: Air is leaking out of an inflated balloon in the shape of a sphere at a rate of $230 \pi$ cubic centimeters per minute. At the instant when the radius is 4 centimeters, what is the rate of change of the radius of the balloon?

EX \#3: A stone is dropped into a calm body of water, causing ripples in the form of concentric circles. The radius of the outer ripple is increasing at a rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area of the disturbed water changing?

EX \#4: Water is leaking out of a cylindrical tank at a rate of 3 cubic feet per second. If the radius of the tank is 4 feet, at what rate is the depth of the water changing at any instant during the leak?

EX \#5: A cone has a diameter of 10 inches and a height of 15 inches. Water is being poured into the cone so that the height of the water level is changing at a rate of 1.2 inches per second. At the instant when the radius of the expose surface area of the water is 2 inches, at what rate is the volume of the water changing?

## EX \#6:

A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second.
a. How fast is the top of the ladder moving down the wall when the base of the ladder is 7 feet from the wall?
b. Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.
c. Find the rate at which the angle formed by the ladder and the wall of the house is changing when the base of the ladder is 9 feet from the wall.

EX \#7: The radius of a sphere is increasing at a rate of 2 inches per minute. Find the rate of change of the surface area of the sphere when the radius is 6 inches.

EX \#8: A spherical balloon is expanding at a rate of $60 \pi$ cubic inches per second. How fast is the surface area of the balloon expanding when the radius of the balloon is 4 inches.

EX \#7: An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna. When the plane is 10 miles past the antenna, the rate at which the distance between the antenna and the plane is changing is 240 miles per hour. What is the speed of the plane?

## I can answer questions that relate position, velocity \& acceleration

## EX \#1:

A particle moves along a horizontal line so that its position at any time $t \geq 0$ is given by $s(t)=2 t^{3}-7 t^{2}+4 t+5$, where $s$ is measured in meters and $t$ in seconds.
(a) Find the velocity at time $t$ and at $t=1$ second.
(b) When is the particle at rest? Moving left? Moving right? Justify your answers.
(c) Find the acceleration at time $t$ and at $t=1$ seconds.
(d) Find the displacement of the particle between $t=0$ and $t=3$ seconds. Explain the meaning of your answer.

## Ex. (continued)

A particle moves along a horizontal line so that its position at any time $t \geq 0$ is given by $s(t)=2 t^{3}-7 t^{2}+4 t+5$, where $s$ is measured in meters and $t$ in seconds.
(e) Find the distance traveled by the particle between $t=0$ and $t=3$ seconds.
(f) When is the particle speeding up? Slowing down? Justify your answer.
(Hint: Since speed is the absolute value of velocity, the particle is:

1) Speeding up when the velocity and acceleration have the same sign (both pos. or both neg.)
2) Slowing down when the velocity and acceleration have opposite signs (one pos. and one neg.)

EX \#2: A particle moves along a horizontal line so that its position at any time $t \geq 0$ is given by $s(t)=t^{3}-5 t^{2}+2 t-3$
(a) When is the particle moving right? moving left? Justify your answer.
(b) Find the distance traveled by the particle from $t=0$ to $t=5$. Justify your answer.
(c) Find the intervals where the speed is increasing. Justify your answer.

EX \#3: . (Multiple Choice) A particle moves along the $x$-axis so that at any time $t \geq 0$, its velocity is given by $v(t)=3+4.1 \cos (0.9 t)$. What is the acceleration of the particle at time $t=4$ ? CALC ACTIVE
(A) -2.016
(B) -0.677
(C) 1.633
(D) 1.814
(E) 2.978

## AP Quick Sheet--Particle Motion

These are the following concepts and formulas that will be needed for Particle Motion on the AP Calculus Exam.

- All time intervals below are $a \leq t \leq b$
- Total Distance =for now you need to find and add the height difference between each point Note: Initial position is NOT used! Later we will use the formula


## TBAшнш

- Position $=$

- $\quad$ Speed $=|v(t)| \quad$ Note: This concept is not used often on the AB exam
- Speed Inc/Dec at $\mathrm{t}=\mathrm{c}$ : Check BOTH v(c) AND a(c).

If $\mathrm{v}(\mathrm{c})$ and $\mathrm{a}(\mathrm{c})$ have SAME sign $\rightarrow$ Speed is INCREASING

If $\mathrm{v}(\mathrm{c})$ and $\mathrm{a}(\mathrm{c})$ have OPPOSITE sign $\rightarrow$ Speed is DECREASING

- Particle moving Right/Up : $\mathrm{v}(\mathrm{t})>0$
- Particle moving Left/Down: $\mathrm{v}(\mathrm{t})<0$
- Particle Changes Direction: Set $v(t)=0$ and look for sign change in $v(t)$
- Average Value of $f(x)$ TBA.....
- Average Rate of $\mathrm{f}(\mathrm{x})=\frac{f(b)-f(a)}{b-a}$

Important Notes for Average Formulas: If you are wanting the average of the equation that is given in the problem, use the AVERAGE VALUE formula.

If you want the average of the rate/derivative of the given equation, use the AVERAGE RATE formula.

For example: Most particle problems give the velocity equation. If they ask for average velocity, use the AVERAGE VALUE formula, but if they ask for average acceleration, use the AVERAGE RATE formula.

## Particle Motion Reference Guide

Typically, if a particle is moving along the x -axis at any time, $t, x(t)$ represents the position of the particle; along the $y$-axis, $y(t)$ is often used; along another straight line, $s(t)$ is often used. In addition, $v(t)$ is typically used to represent the velocity of the particle. In these types of particle motion problems,

- Position: $x(t)$ or $s(t)$

Velocity: $v(t)=s^{\prime}(t)$
Acceleration: $a(t)=v^{\prime}(t)=s^{\prime \prime}(t)$

- "Initially" means when time $t=0$.
- "At the origin" means $x(t)=0$.
- "At rest" means velocity $v(t)=0$.
- Average velocity of the particle is $\frac{\Delta s}{\Delta t}=\frac{s\left(t_{2}\right)-s\left(t_{1}\right)}{t_{2}-t_{1}}$ when the position function is given;

Average velocity of the particle is
TBA......

- Average acceleration of the particle is $\frac{\Delta v}{\Delta t}=\frac{v\left(t_{2}\right)-v\left(t_{1}\right)}{t_{2}-t_{1}}$ when the velocity function is given; Average acceleration of the particle is TBA..... function is given.
- If the velocity of the particle is positive, then the particle is moving to the right (or up).
- If the velocity of the particle is negative, then the particle is moving to the left (or down).
- If the acceleration of the particle is positive, then the velocity is increasing.
- If the acceleration of the particle is negative, then the velocity is decreasing.
- If the velocity of the particle is negative, then the particle is moving to the left (or down).
- If the acceleration of the particle is positive, then the velocity is increasing.
- If the acceleration of the particle is negative, then the velocity is decreasing.
- Speed is the absolute value of velocity.
- If the velocity and acceleration have the same sign (both positive or both negative), then speed is increasing.
- If the velocity and acceleration are opposite in sign (one is positive and the other is negative), then speed is decreasing.
- To determine total distance traveled over a time interval, you must calculate the sum of the absolute values of the differences in position between all resting points or calculate the area under the absolute value of the velocity curve,
- Displacement can be determined using displacement $=$


