AP Type Questions 1: Rate and Accumulation

These questions are often in context with a lot of words describing a situation in which some things are changing. There are usually two rates acting in opposite ways (sometimes called in-out question). Students are asked about the change that the rates produce over some time interval either separately or together.

The rates are often fairly complicated functions. If they are on the calculator allowed section, students should store the functions in the equation editor of their calculator and use their calculator to do any graphing, integration, or differentiation that may be necessary.

The main idea is that over the time interval [*a*, *b*] the integral of a rate of change is the net amount of change

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

If the question asks for an amount, look around for a rate to integrate.

The final (accumulated) amount is the initial amount plus the accumulated change:

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt,$$

where x_0 is the initial time, and $f(x_0)$ is the initial amount. Since this is one of the main interpretations of the definite integral the concept may come up in a variety of situations.

What students should be able to do:

- Be ready to read and apply; often these problems contain a lot of writing which needs to be carefully read.
- Recognize that rate = derivative.
- Recognize a rate from the units given without the words “rate” or “derivative.”
- Find the change in an amount by integrating the rate. The integral of a rate of change gives the amount of change (FTC):

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

- Find the final amount by adding the initial amount to the amount found by integrating the rate. If $x = x_0$ is the initial time, and $f(x_0)$ is the initial amount, then final accumulated amount is

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt,$$

- Understand the question. It is often not necessary to as much computation as it seems at first.
- Use FTC to differentiate a function defined by an integral.
- Explain the meaning of a derivative or its value in terms of the context of the problem. The explanation should contain (1) what it represents, (2) its units, and (3) how numerical argument applies in context.
- Explain the meaning of a definite integral or its value in terms of the context of the problem. The explanation should contain (1) what it represents, (2) its units, and (3) how the limits of integration apply in context.
- Store functions in their calculator recall them to do computations on their calculator.
- If the rates are given in a table, be ready to approximate an integral using a Riemann sum or by trapezoids.
- Do a max/min or increasing/decreasing analysis.

Shorter questions on this concept appear in the multiple-choice sections. As always, look over as many questions of this kind from past exams as you can find.

Typical free-response examples:

- 2013 AB1/BC1,
- 2015 AB1/BC1
- 2016 AB 1.
- One of my favorites Good Question 6 (2002 AB 4)

Typical multiple-choice examples from non-secure exams:

- 2012 AB 8, 81, 89
- 2012 BC 8 (same as AB 8)

Type 2 Questions: Linear Motion

"A particle (or car, or bicycle) moves on a number line …."

These questions may give the position equation, the velocity equation (most often), or the acceleration equation of something that is moving, along with an initial condition. The questions ask for information about motion of the particle: its direction, when it changes direction, its maximum position in one direction (farthest left or right), its speed, etc.

The particle may be a "particle," a person, car, a rocket, etc. Particles don't really move in this way, so the equation or graph should be considered to be a model. The question is a versatile way to test a variety of calculus concepts since the position, velocity, or acceleration may be given as an equation, a graph, or a table; be sure to use examples of all three forms during the review.

Many of the concepts related to motion problems are the same as those related to function and graph analysis (Type 3). Stress the similarities and show students how the same concepts go by different names. For example, finding when a particle is "farthest right" is the same as finding the when a function reaches its "absolute maximum value."

Function	***Linear Motion***
Value of a function at x	*position at time t*
First derivative	*velocity*
Second derivative	*acceleration*
Increasing	*moving to the right or up*
Decreasing	*moving to the left or down*
Absolute Maximum	*farthest right*
Absolute Minimum	*farthest left*
y' = 0	*"at rest"*
y' changes sign	*object changes direction*
Increasing & cc up	*speed is increasing*
Increasing & cc down	*speed is decreasing*
Decreasing & cc up	*speed is decreasing*
Decreasing & cc down	*speed is increasing*
Speed	*absolute value of velocity*

There is usually one free-response question and three or more multiple-choice questions on this topic.

The position, $s(t)$, is a function of time. The relationships are:

- The velocity is the derivative of the position, $s'(t) = v(t)$. Velocity is has direction (indicated by its sign) and magnitude. Technically, velocity is a vector; the term "vector" will not appear on the AB exam.
- Speed is the absolute value of velocity; it is a number, not a vector.
- Acceleration is the derivative of velocity and the second derivative of position, $a(t) = v'(t) = s''(t)$. It, too, has direction and magnitude and is a vector.
- Velocity is the antiderivative of the acceleration
- Position is the antiderivative of velocity.

What students should be able to do:

- Understand and use the relationships above.
- Distinguish between position at some time and the total distance traveled during the time period.
- The total distance traveled is the definite integral of the speed (absolute value of velocity) $\int_a^b |v(t)|\,dt$.
- The net distance traveled, displacement, is the definite integral of the velocity (rate of change): $\int_a^b v(t)\,dt$. Note that "displacement" has not been used preciously on AP exam, but (as per the new *Course and Exam Description*) may be used now. Be sure your students know this term.
- The final position is the initial position plus the *displacement* (definite integral of the rate of change from $x = a$ to $x = t$): $s(t) = s(a) + \int_a^t v(x)\,dx$ Notice that this is an *accumulation* function equation (Type 1).
- Initial value differential equation problems: given the velocity or acceleration with initial condition(s) find the position or velocity. These are easily handled with the accumulation equation in the bullet above.
- Find the speed at a given time. The speed is the absolute value of the velocity.
- Find average speed, velocity, or acceleration
- Determine if the speed is increasing or decreasing.
  - If at some time, the velocity and acceleration have the *same* sign then the speed is increasing.If they have *different* signs the speed is decreasing.
  - ➢ If the velocity graph is moving away from (towards) the t-axis the speed is increasing (decreasing). Picture a car moving along a road going forwards (in the positive direction) its velocity is positive.

    - ➢ If you step on the gas your, acceleration pulls you in the direction you are moving and your speed increases. ($v > 0$, $a > 0$, speed increases)
    - ➢ Going too fast is not good, so you put on your breaks, you now accelerate in the opposite direction (decelerate?), but you are still moving forward, but slower. ($v > 0$, $a < 0$, speed decreases)
    - ➢ Finally you stop. Then you shift into reverse and start moving backwards (negative velocity) and you push on the gas to accelerate in the negative direction, so your speed increases. ($v < 0$, $a < 0$, speed increases)
    - ➢ Then you put on the breaks (accelerate in the positive direction) and your speed decreases again. ($v < 0$, $a > 0$, speed decreases)
- Use a difference quotient to approximate derivative.
- Riemann sum approximations.
- Units of measure.
- Interpret meaning of a derivative or a definite integral in context of the problem

Shorter questions on this concept appear in the multiple-choice sections. As always, look over as many questions of this kind from past exams as you can find.

Free-response examples:

- Equation stem 2017 AB 5,
- Graph stem: 2009 AB1/BC1,
- Table stem 2015 AB 3/BC3

Multiple-choice examples from non-secure exams:

- 2012 AB 6, 16, 28, 79, 83, 89
- 2012 BC 2, 89

Type 3 Questions: Graph and Function Analysis

The long name is "Here's the graph of the derivative, tell me things about the function."
Students are given either the equation of the derivative of a function or a graph identified as the derivative of a function but no equation is given. It is not expected that students will write the equation (although this may be possible); rather, students are expected to determine important features of the function directly from the graph of the derivative. They may be asked for the location of extreme values, intervals where the function is increasing or decreasing, concavity, *etc.* They may be asked for function values at points.
The graph may be given in context and student will be asked about that context. The graph may be identified as the velocity of a moving object and questions will be asked about the motion.
Less often the function's graph may be given and students will be asked about its derivatives.

What students should be able to do:

- Read information about the function from the graph of the derivative. This may be approached by derivative techniques or by antiderivative techniques.
- Find and justify where the function is increasing or decreasing.
- Find and justify extreme values (1st and 2nd derivative tests, Closed interval test aka. Candidates' test).
- Find and justify points of inflection.
- Find slopes (second derivatives, acceleration) from the graph.
- Write an equation of a tangent line.
- Evaluate Riemann sums from geometry of the graph only.
- FTC: Evaluate integral from the area of regions on the graph.
- FTC: The function, $g(x)$, maybe defined by an integral where the given graph is the graph of the integrand, $f(t)$, so students should know that if, $g(x) = g(a) + \int_a^t f(t)\,dt$ then $g'(x) = f(x)$ and $g''(x) = f'(x)$. In this case students should write $g'(t) = f(t)$ on their answer paper, so it is clear to the reader that they understand this.

Not only must students be able to identify these things, but they are usually asked to justify their answer and reasoning. See the JUSTIFICATION DOCUMENT for more details.
The ideas and concepts that can be tested with this type question are numerous. The type appears on the multiple-choice exams as well as the free-response. Between multiple-choice and free-response this topic may account for 15% or more of the points available on recent tests. It is very important that students are familiar with all the ins and outs of this situation.

As with other questions, the topics tested come from the entire year's work, not just a single unit. Study past exams; look them over and see the different things that can be asked.

For some previous posts on this subject see October 15, 17, 19, 24, 26 (my most read post), 2012 and January 25, 28, 2013
Free-response questions:

- Function given as a graph, questions about its integral (so by FTC the graph is the derivative): 2014 AB 3/BC 3
- Function given as an equation: 2016 AB 6
- Function given as a graph 2016 AB 3/BC 3
- Table and graph of function given, questions about related functions: 2017 AB 6,
- Derivative given as a graph: 2016 AB 5 and 2017 AB 3

Multiple-choice questions from non-secure exam. Notice the number of questions all from the same year; this is in addition to one free-response question (~25 points on AB and ~23 points on BC out of 108 points total)

- 2012 AB: 2, 5, 15, 17, 21, 22, 24, 26, 76, 78, 80, 83, 82, 84, 85, 87
- 2012 BC 3, 11, 12, 15, 12, 18, 21, 76, 78, 80, 81, 84, 88, 89

Type 4 Questions: Area and Volume Problems

Given equations that define a region in the plane students are asked to find its area, the volume of the solid formed when the region is revolved around a line, and/or the region is used as a base of a solid with regular cross-sections. This standard application of the integral has appeared every year since year one (1969) on the AB exam and almost every year on the BC exam. You can be pretty sure that if a free-response question on areas and volumes does not appear, the topic will be tested on the multiple-choice section.

What students should be able to do:

- Find the intersection(s) of the graphs and use them as limits of integration (calculator equation solving). Write the equation followed by the solution; showing work is not required. Usually no credit is earned until the solution is used in context (as a limit of integration). Students should know how to store and recall these values to save time and avoid copy errors.
- Find the area of the region between the graph and the *x*-axis or between two graphs.
- Find the volume when the region is revolved around a line, not necessarily an axis or an edge of the region, by the disk/washer method.
- Find the volume of a solid with regular cross-sections whose base is the region between the curves. For an interesting variation on this idea see 2009 AB 4(b)
- Find the equation of a vertical line that divides the region in half (area or volume). This involves setting up an integral equation where the limit is the variable for which the equation is solved.

If this question appears on the calculator active section, it is expected that the definite integrals will be evaluated on a calculator. Students should write the definite integral with limits on their paper and put its value after it. It is *not* required to give the antiderivative and if a student gives an incorrect antiderivative they will lose credit even if the final answer is (somehow) correct. There is a calculator program available that will give the set-up and not just the answer so recently this question has been on the no calculator allowed section. (The good news is that in this case the integrals will be easy or they will be set-up-but-do-not-integrate questions.)

Occasionally, other type questions have been included as a part of this question. See 2016 AB5/BC5 which included an average value question and a related rate question along with finding the volume.

Shorter questions on this concept appear in the multiple-choice sections. As always, look over as many questions of this kind from past exams as you can find.

Free-response questions:

- 2014 AB 2, 2013 AB 5.
- 2015 AB 2
- Variations: 2009 AB 4,
- 2016 AB5/BC5,
- 2017 AB 1 (using a table),
- Perimeter 2011 BC 3 and 2014 BC 5

Multiple-choice questions from non-secure exams:

- 2008 AB 83 (Use absolute value),
- 2012 AB 10, 92
- 2012 BC 87, 92 (Polar area)

Type 5: Table and Riemann Sum Questions

Tables may be used to test a variety of ideas in calculus including analysis of functions, accumulation, theory and theorems, and position-velocity-acceleration, among others. Numbers and working with numbers is part of the Rule of Four and table problems are one way this is tested. This question often includes an equation in a latter part of the problem that refers to the same situation.

What students should be able to do:

- Find the average rate of change over an interval
- Approximate the derivative using a difference quotient. Use the two values closest to the number at which you are approximating. This amounts to finding the slope or rate of change. Show the quotient even if you can do the arithmetic in your head.
- Use Riemann sums (left, right, midpoint), or a trapezoidal approximation to approximate the value of a definite integral using values in the table (typically with uneven subintervals). The Trapezoidal Rule, *per se*, is not required; it is expected that students will add the areas of a small number of trapezoids without reference to a formula.
- Average value, average rate of change, Rolle's theorem, the Mean Value Theorem and the Intermediate Value Theorem. (See 2007 AB 3 – four simple parts that could be multiple-choice questions; the mean on this question was 0.96 out of a possible 9.)
- These questions are usually presented in some context and answers should be in that context.
- Unit analysis.

Do's and Don'ts

Do: Remember that you do not know what happens between the values in the table unless some other information is given. For example, **do not** assume that the largest number in the table is the maximum value of the function, or that the function is decreasing just because a value is less than the preceding value.

Do: Show what you are doing even if you can do it in your head. If you're finding a slope, show the quotient.

Do Not do arithmetic: A long expression consisting entirely of numbers such as you get when doing a Riemann sum, does not need to be simplified in any way. If you a simplify correct answer incorrectly, you will lose credit.

Do Not leave expression such as R(3) – pull its numerical value from the table.

Do Not: Find a regression equation and then use that to answer parts of the question. While regression is perfectly good mathematics, regression equations are not one of the four things students may do with their calculator. Regression gives only an approximation of our function. The exam is testing whether students can work with numbers.

Free-response examples:

- 2007 AB 3 (4 simple parts on various theorems, yet the mean score was 0.96 out of 9),
- 2017 AB 1/BC 1, and AB 6,
- 2016 AB 1/BC 1

Multiple-choice questions from non-secure exams:

- 2012 AB 8, 86, 91
- 2012 BC 8, 81, 86 (81 and 86 are the same on both the AB and BC exams)

Type 6 Questions: Differential Equations

Differential equations are tested every year. The actual solving of the differential equation is usually the main part of the problem, but it is accompanied by a related question such as a slope field or a tangent line approximation. BC students may also be asked to approximate using Euler's Method. Large parts of the BC questions are often suitable for AB students and contribute to the AB sub-score of the BC exam.

What students should be able to do

- Find the *general solution* of a differential equation using the method of separation of variables (this is the *only* method tested).
- Find a *particular solution* using the initial condition to evaluate the constant of integration – initial value problem (IVP).
- **NEW** Determine the domain restrictions on the solution of a differential equation. See the following:
    - The interval must be open since derivatives are not defined at the endpoints of intervals. The derivative is a two-sided limit and at an endpoint you can only approach from one side. While one-sided derivatives may be defined, they are a more burdensome requirement and not necessary.
    - Practically speaking, this means that the solution may not cross a vertical asymptote or go through a point where the function is undefined; it must stay on the side where the initial condition is.
    - The differential equation must be true in the sense that substituting the solution and its derivative(s) into the differential equation must result in an identity. (See 2007 AB 4(b) for practice).
    - And of course, the initial condition point's *x*-coordinate must be in the domain
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- Understand that proposed solution of a differential equation is a function (not a number) and if it and its derivative are substituted into the given differential equation the resulting equation is true. This may be part of doing the problem even if solving the differential equation is not required (see 2002 BC 5 – parts a, b and d are suitable for AB)
- Growth-decay problems.
- Draw a slope field by hand.
- Sketch a *particular solution* on a given slope field.
- Interpret a slope field.
- Multiple-choice: Given a differential equation, identify is slope field.
- Multiple-choice: Given a slope field identify its differential equation.
- Use the given derivative to analyze a function such as finding extreme values
- For BC only: Use Euler's Method to approximate a solution.
- For BC only: use the method of partial fractions to find the antiderivative after separating the variables.
- For BC only: understand the logistic growth model, its asymptotes, meaning, etc. The exams so far, have never asked students to actually solve a logistic equation IVP

Look at the scoring standards to learn how the solution of the differential equation is scored, and therefore, how students should present their answer. This is usually the one free-response answer with the most points riding on it. Starting in 2016 the scoring has changed slightly. The five points are now distributed this way:

- one point for separating the variables
- one point each for finding the antiderivatives
- one point for including the constant of integration and using the initial condition – that is, for writing "+ C" on the paper with one of the antiderivatives and substituting the initial condition; finding the value of *C* is included in the "answer point." and
- one point for solving for *y*: the "answer point", for the correct answer. This point includes all the algebra and arithmetic in the problem including solving for *C*..

In the past, the domain of the solution is often included on the scoring standard, but unless it is specifically asked for in the question students do not need to include it. However, the new CED lists "EK 3.5A3 Solutions to differential equations may be subject to domain restrictions." Perhaps this will be asked in the future.
Shorter questions on this concept appear in the multiple-choice sections. As always, look over as many questions of this kind from past exams as you can find.

Free-response examples:

- 2017 AB4/BC4,
- 2016 AB 4, BC 4, (different questions)
- 2015 AB4/BC4,
- 2013 BC 5
- and a favorite Good Question 2 and Good Question 2 Continued

Multiple-choice examples from non-secure exams:

- 2012 AB 23, 25
- 2012 BC: 12, 14, 16, 23

Type 7 Questions: Miscellaneous

Any topic in the Course and Exam Description may be the subject of a free-response or multiple-choice question. There are topics that are not asked often enough to be classified as a type of their own. The two topics listed here have been the subject of full free-response questions or major parts of them. Other topics occasionally asked are mentioned in the question list at the end of the post.

Implicitly defined relations and implicit differentiation
These questions may ask students to find the first or second derivative of an implicitly defined relation. Often the derivative is given and students are required to show that it is correct. (This is because without the correct derivative the rest of the question cannot be done.) The follow-up is to answer questions about the function such as finding an extreme value, second derivative test, or find where the tangent is horizontal or vertical.

What students should know how to do

- Know how to find the first derivative of an implicit relation using the product rule, quotient rule, chain rule, etc.
- Know how to find the second derivative, including substituting for the first derivative.
- Know how to evaluate the first and second derivative by substituting both coordinates of a given point. (Note: If all that is needed is the numerical value of the derivative then the substitution is often easier if done before solving for dy/dx or d^2y/dx^2, and as usual the arithmetic need not be done.)
- Analyze the derivative to determine where the relation has horizontal and/or vertical tangents.
- Write and work with lines tangent to the relation.
- Find extreme values. It may also be necessary to show that the point where the derivative is zero is actually on the graph and to justify the answer.

Simpler questions about implicit differentiation my appear on the multiple-choice sections of the exam.

Related Rates
Derivatives are rates and when more than one variable is changing over time the relationships among the rates can be found by differentiating with respect to time. The time variable may not appear in the equations. These questions appear occasionally on the free-response sections; if not there, then a simpler version may appear in the multiple-choice sections. In the free-response sections they may be an entire problem, but more often appear as one or two parts of a longer question.

What students should know how to do

- Set up and solve related rate problems.
- Be familiar with the standard type of related rate situations, but also be able to adapt to different contexts.
- Know how to differentiate with respect to *time. T*hat is, find dy/dt even if there is no time variable in the given equations using any of the differentiation techniques.
- Interpret the answer in the context of the problem.
- Unit analysis.

Shorter questions on this concept also appear in the multiple-choice sections. As always, look over as many questions of this kind from past exams as you can find.